## Time-Dependent Setup

We have a time-indep. $\hat{H}_0$, w/

$$\hat{H}_0|\psi_j\rangle = E_j|\psi_j\rangle \qquad j = 1 \to \infty$$

we have a t-dep. "perturbation" $\hat{H}'(t)$, & we want to solve:

$$(\hat{H}_0 + \hat{H}')|\psi\rangle = i\hbar \frac{d}{dt}|\psi\rangle \qquad (*)$$

Any $|\psi\rangle$ can be written as:

$$|\psi\rangle = \sum_{j=1}^{\infty} c_j(t) e^{-iE_jt/\hbar}|\psi_j\rangle$$

& running this through (*) gave:

$$\sum_{j=1}^{\infty} c_j(t) e^{-iE_jt/\hbar}\hat{H}'|\psi_j\rangle = i\hbar\sum_{j=1}^{\infty}\dot{c}_j(t)e^{-iE_jt/\hbar}|\psi_j\rangle$$

hit both sides w/ $\langle\psi_k|$ & use $\langle\psi_k|\psi_j\rangle = \delta_{jk}$ to get:

$$\sum_{j=1}^{\infty} c_j(t)e^{-iE_jt/\hbar}\underbrace{\langle\psi_k|\hat{H}'|\psi_j\rangle}_{\equiv H'_{kj}} = i\hbar\dot{c}_k(t)e^{-iE_kt/\hbar}$$

we can write the eqn. in ∞-dim. matrix form:

$$\frac{d}{dt}\begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix} = \frac{1}{i\hbar}\begin{pmatrix} H'_{11} & H'_{12}e^{+i(E_1-E_2)t/\hbar} & H'_{13}e^{+i(E_1-E_3)t/\hbar} & \cdots \\ H'_{21}e^{+i(E_2-E_1)t/\hbar} & H'_{22} & H'_{23}e^{+i(E_2-E_3)t/\hbar} & \cdots \\ & & & \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix}.$$

focus on 2-states – a "2-level system" – the upper 2x2 block, say. – label the states a & b (we'll need # indices later on)

$$\frac{d}{dt}\begin{pmatrix} c_a \\ c_b \end{pmatrix} = \frac{1}{i\hbar}\begin{pmatrix} H'_{aa} & H'_{ab}e^{-i(E_b-E_a)t/\hbar} \\ H'_{ba}e^{+i(E_b-E_a)t/\hbar} & H'_{bb} \end{pmatrix}\begin{pmatrix} c_a \\ c_b \end{pmatrix}$$

define the freq. $\omega_0 \equiv \frac{E_b - E_a}{\hbar}$, & note that $H'_{ba} = (H'_{ab})^*$ so the matrix here is Hermitian.

If the matrix above was t-indep. (or if other conditions held) we could solve for $c_a(t)$ & $c_b(t)$ via:

$$\begin{pmatrix} c_a(t) \\ c_b(t) \end{pmatrix} = e^{\int_0^t \mathbb{H}(\bar{t})d\bar{t}}\begin{pmatrix} c_a(0) \\ c_b(0) \end{pmatrix}.$$

but alas...

The interesting action occurs in the off-diagonal terms – assume for now, that $H'_{aa} = H'_{bb} = 0$.

We have:

$$\dot{c}_a(t) = -\frac{i}{\hbar}\overset{\text{t-dep.}}{H'_{ab}}e^{-i\omega_0 t}c_b(t)$$

$$\dot{c}_b(t) = -\frac{i}{\hbar}H'_{ba}e^{i\omega_0 t}c_a(t)$$

we have a formal solution here that is similar to our scattering one – integrate the top eqn. in time to get:

$$c_a(t) = -\frac{i}{\hbar}\int_0^t H'_{ab}(\bar{t})e^{-i\omega_0\bar{t}}\underbrace{c_b(\bar{t})}_{}d\bar{t}$$

need to know $c_b(t)$ to carry out the integral.

## Perturbation Setup

For $H'_{ab} = \epsilon \bar{H}_{ab}$ w/ $\epsilon \ll 1$, expand the coefficients:

$$c_a(t) = c_a^0(t) + \epsilon c_a^1(t) + \cdots$$

$$c_b(t) = c_b^0(t) + \epsilon c_b^1(t) + \cdots$$

Now the $c_a$-eqn. reads:

$$(\dot{c}_a^0 + \epsilon \dot{c}_a^1 + \cdots) = -\frac{i}{\hbar} \epsilon \bar{H}_{ab} e^{-i\omega_0 t} (c_b^0 + \epsilon c_b^1 + \cdots)$$

& collect in powers of $\epsilon$:

$$\epsilon^0 : \ \dot{c}_a^0 = 0 \Rightarrow c_a^0 = k_a \leftarrow \text{const.}$$

$$\epsilon^1 : \ \dot{c}_a^1 = -\frac{i}{\hbar} \bar{H}_{ab} e^{-i\omega_0 t} c_b^0$$

& sim. for the $c_b(t)$ eqn.

$$\epsilon^0 : \ \dot{c}_b^0 = 0 \Rightarrow c_b^0 = k_b$$

$$\dot{c}_b^1 = -\frac{i}{\hbar} \bar{H}_{ba} e^{i\omega_0 t} c_a^0$$

so:

known, const. here

$$c_a^1(t) = -\frac{i}{\hbar} \int_0^t \bar{H}_{ab}(\bar{t}) e^{-i\omega_0 \bar{t}} c_b^0 \, d\bar{t} \qquad (1)$$

$$c_b^1(t) = -\frac{i}{\hbar} \int_0^t \bar{H}_{ba}(\bar{t}) e^{i\omega_0 \bar{t}} c_a^0 \, d\bar{t} \qquad (2)$$

In general:

known

$$c_a^j(t) = -\frac{i}{\hbar} \int_0^t \bar{H}_{ab}(\bar{t}) e^{-i\omega_0 \bar{t}} c_b^{j-1}(\bar{t}) \, d\bar{t}$$

you can write out the iterations as in the scattering case:

$$c_a^2(t) = -\frac{i}{\hbar} \int_0^t e^{-i\omega_0 \bar{t}} \bar{H}_{ab}(\bar{t}) c_b^1(\bar{t}) \, d\bar{t}$$

$$= \left(-\frac{i}{\hbar}\right)^2 \int_0^t e^{-i\omega_0 \bar{t}} \bar{H}_{ab}(\bar{t}) \left[ \int_0^{\bar{t}} e^{i\omega_0 \bar{\bar{t}}} \bar{H}_{ba}(\bar{\bar{t}}) \underbrace{c_a^0(\bar{\bar{t}})}_{\text{const}} d\bar{\bar{t}} \right] d\bar{t}$$

## Convention

We'll work out the 1st order perturbation theory result — assume:

$$c_a^0(0) = 1 \qquad c_b^0(0) = 0 \qquad (\text{we start in } |\psi_a\rangle)$$

and from (1), we have: $c_a^1 = 0$, then (2) is:

$$c_b^1 = -\frac{i}{\hbar} \int_0^t \bar{H}_{ba}(\bar{t}) e^{i\omega_0 \bar{t}} \, d\bar{t}$$

so our approximation is:

$$c_a(t) = c_a^0(t) + \epsilon c_a^1(t) = 1$$

$$c_b(t) = c_b^0(t) + \epsilon c_a^2(t) = -\frac{i}{\hbar} \int_0^t H_{ba}(\bar{t}) e^{i\omega_0 \bar{t}} \, d\bar{t}$$

A typical perturbing potential is one of the form:

$$\hat{H}' = V(\vec{r}) \cos(\omega t)$$

w/ $H'_{ba} = \underbrace{\langle \psi_b | V | \psi_a \rangle}_{\equiv V_{ba}} \cos(\omega t)$

then $c_b(t) = -\frac{i}{\hbar}\int_0^t V_{ba}\, e^{i\omega_0 \bar{t}} \cos(\omega \bar{t})\, d\bar{t}$

$$= -\frac{i}{\hbar}\frac{V_{ba}}{2}\int_0^t \left[e^{i(\omega_0+\omega)\bar{t}} + e^{i(\omega_0-\omega)\bar{t}}\right] d\bar{t}$$

$$= -\frac{iV_{ba}}{2\hbar}\left[\frac{e^{i(\omega_0+\omega)\bar{t}}}{i(\omega_0+\omega)}\Big|_0^t + \frac{e^{i(\omega_0-\omega)\bar{t}}}{i(\omega_0-\omega)}\Big|_0^t\right]$$

assume $\omega \approx \omega_0$ - driving near the transition freq so the second term dominates:

$$c_b(t) \approx -\frac{V_{ba}}{2\hbar}\, e^{i(\omega_0-\omega)t/2}\left[\frac{e^{i(\omega_0-\omega)t/2} - e^{-i(\omega_0-\omega)t/2}}{\omega_0-\omega}\right]$$

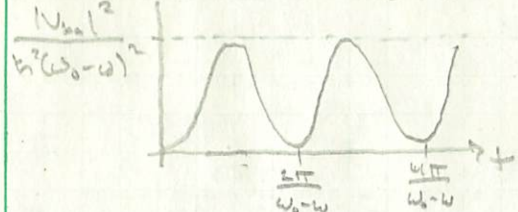
$$= -\frac{iV_{ba}}{\hbar}\, e^{i(\omega_0-\omega)t/2}\left[\frac{\sin((\omega_0-\omega)t/2)}{(\omega_0-\omega)}\right]$$

The probability of finding the particle in $|\psi_b\rangle$ is, then:

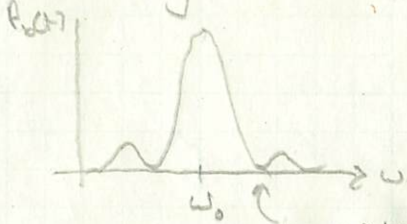
$$P_b(t) = |c_b(t)|^2 = \frac{|V_{ba}|^2}{\hbar^2}\frac{\sin^2((\omega_0-\omega)t/2)}{(\omega_0-\omega)^2}$$

as a function of time, $P_b$ is periodic w/ period:

$$T = \frac{2\pi}{\omega_0-\omega}$$




$P_b(t)$ is sharply peaked in freq.:




$(\omega-\omega_0)t/2 = \pi \Rightarrow \omega = \omega_0 + 2\pi/t$

Whence the oscillatory perturbation?