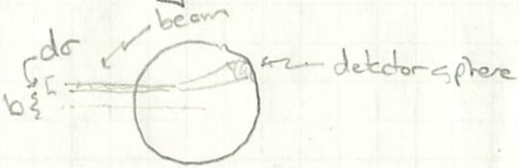
Scattering in D=3



The number of particles passing through a $d\sigma$ cross-section in time Δt is:

$$N = \mathcal{L}\, d\sigma$$

That same number goes into the area patch $k^2 d\Omega$ on the detector

$$N = \mathcal{L}' k^2 d\Omega$$

then

$$d\sigma = \underbrace{\frac{\mathcal{L}'}{\mathcal{L}} k^2}_{\equiv D(\Theta)} d\Omega$$

w/ $D(\Theta)$ the "differential scattering cross section."

The beam cross-section, $d\sigma$ can be written in terms of b & db:

$$d\sigma = b\, d\phi\, db$$

& $db = \frac{db}{d\Theta} d\Theta$

so $d\sigma = b \left|\frac{db}{d\Theta}\right| d\Theta\, d\phi = \underbrace{\frac{b}{\sin\Theta}\left|\frac{db}{d\Theta}\right|}_{= D(\Theta)} \underbrace{\sin\Theta\, d\Theta\, d\phi}_{d\Omega}$

$$D(\Theta) = \frac{b}{\sin\Theta}\left|\frac{db}{d\Theta}\right|$$

example - for hard sphere scattering,

$$b = R\cos(\Theta/2) \quad , \quad \frac{db}{d\Theta} = -\tfrac{1}{2} R \sin(\Theta/2).$$

$$D(\Theta) = \frac{+\tfrac{1}{2}R^2 \cos(\Theta/2)|\sin(\Theta/2)|}{\sin\Theta} = \frac{1}{4}R^2 \quad \text{(uniform)}$$

the "total cross section" is

$$\sigma = \int_0^{2\pi}\int_0^{\pi} \frac{b}{\sin\Theta}\left|\frac{db}{d\Theta}\right| \sin\Theta\, d\Theta\, d\phi.$$

$$= \pi R^2 \quad ✓$$

Rutherford Scattering

"plum pudding" model of the atom - uniform positive charge w/ e^- floating around.

scattering prediction:

scattering observation:

evidence of a "nucleus"

Classical Scattering

For a potential energy $U(r)$ (spherically symmetric),

$$L = \tfrac{1}{2}mv^2 - U(r) = \tfrac{1}{2}m(\dot r^2 + r^2\dot\theta^2 + r^2\sin^2\theta\,\dot\phi^2) - U(r)$$

Motion occurs in a plane (cons. of ang. mom.), take it to be the x-y plane.

$$\theta = \pi/2 \qquad \dot\theta = 0$$

$$L = \tfrac{1}{2}m(\dot r^2 + r^2\dot\phi^2) - U(r).$$

$$p_r = \frac{\partial L}{\partial \dot r} = m\dot r \qquad p_\phi = \frac{\partial L}{\partial \dot\phi} = mr^2\dot\phi$$

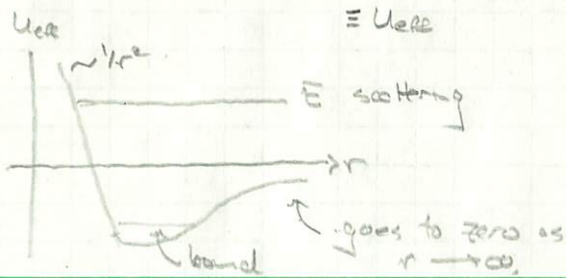
$$H = p_r\dot r + p_\phi\dot\phi - L$$

here $\dot p_\phi = -\dfrac{\partial H}{\partial \phi} = 0$

so p_ϕ is a constant of motion $L_z \equiv p_\phi = mr^2\dot\phi$

& the Hamiltonian becomes:

$$H = \tfrac{1}{2}m\dot r^2 + \underbrace{\frac{L_z^2}{2mr^2} + U(r)}_{\equiv\, U_{eff}}$$



energy is conserved, the Hamiltonian is a constant:

$$E = \tfrac{1}{2}m\dot r^2 + U_{eff}(r)$$

"most" $U(r)$ go like $(1/r)^p$, so let $\rho \equiv 1/r \Rightarrow r = 1/\rho$

$$\dot r = -1/\rho^2\,\dot\rho \Rightarrow E = \tfrac{1}{2}m\dot\rho^2/\rho^4 + U_{eff}(1/\rho)$$

To describe scattering, we only need the shape of the trajectory, not its time-evolution.

parametrize ρ w/ ϕ: $\dfrac{d\rho}{dt} = \dfrac{d\rho}{d\phi}\left(\dfrac{d\phi}{dt}\right) = \dfrac{L_z}{m}\rho^2$

$$= \rho'(\phi)\cdot\frac{L_z}{m}\rho(\phi)^2$$

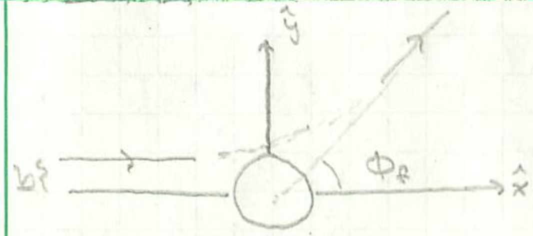
now we have

$$E = \frac{L_z^2}{2m}\rho'(\phi)^2 + U_{eff}(1/\rho(\phi))$$

Take the ϕ-derivative to get a ϕ-parametrized eqn. of motion:

$$\rho''(\phi) = -\frac{m}{L_z^2}\underbrace{\frac{dU_{eff}}{d\phi}}_{=\frac{L_z^2}{m}\rho + \frac{dU}{d\rho}} = -\rho - \frac{m}{L_z^2}\frac{dU}{d\rho}$$

Setup



solve: $\rho''(\phi) = -\rho(\phi) - \frac{m}{L_z^2}\frac{dU}{d\rho}$

w/ initial conditions:

$x = -\infty \qquad \dot{x} = v_0$

$\qquad\qquad\qquad$ at $t \to -\infty$

$y = b \qquad \dot{y} = 0$

In terms of ρ & ϕ, we have:

$\phi = \pi \quad$ w/ $\rho(\pi) = 0$

For $U(r) \xrightarrow[r\to\infty]{} 0$ (typical),

$E = \frac{1}{2} m v_0^2$ is the energy initially,

In terms of r & $\dot{r}$, this is

$E = \frac{1}{2} m \dot{r}(-\infty)^2$

in terms of ρ & ρ', we have

$E = \frac{L_z^2}{2m}\rho'(\pi)^2$

relating the initial $\rho'(\pi)$ to the constant energy.

we also have constant $L_z = (\vec{r} \times \vec{p})_z \big|_{t \to -\infty}$

$\qquad = b m v_0$

w/ $L_z^2 = b^2 m (m v_0^2) = 2 m b^2 E_0$

the full set of "initial conditions" is:

$\phi = \pi : \rho(\pi) = 0 \,, \quad \rho'(\pi)^2 = \frac{2mE_0}{L_z^2} \quad L_z^2 = 2mb^2E_0$

we want to find ϕ_f in terms of b, E_0 & the potential:

$\rho(\phi_f) = 0 \quad$ defines ϕ_f.