Spin-Orbit Coupling

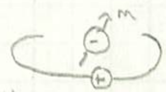
electron has magnetic dipole moment from its intrinsic spin.

2 stories to tell:

1. a moving magnetic dipole "induces" an electric dipole, so the e^-'s $\vec{p}$ interacts w/ the electric field generated by the proton, $\vec{E}$, w/

$$U' = -\vec{p}\cdot\vec{E}$$

2. In the electron's rest frame, the proton moves around, generating a magnetic field, + the energy associated w/ an $\vec{m}$ in a $\vec{B}$ is:



$$U' = -\vec{m}\cdot\vec{B}$$

equivalent views – both require relativistic effects to describe appropriately.

Magnetic Dipole in Motion

A neutral magnetic dipole at rest:

$$m = Ia^2$$

a mechanical model of neutral current:

$$\begin{array}{c} \lambda \\ \hline \end{array} \to v \qquad v \leftarrow \begin{array}{c} -\lambda \\ \hline \end{array} \quad = \quad \xrightarrow[I = 2\lambda v]{}$$

when a line of charge moves, its density "changes"

$$\underbrace{(\quad)}_{\Delta x_0} \;\; \lambda_0 \text{ at rest}$$

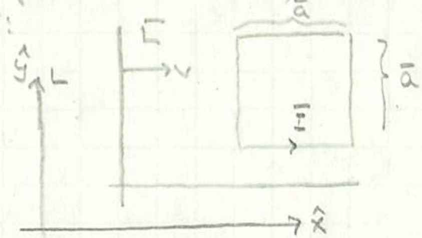
the charge in the interval is $q = \lambda_0 \Delta x_0$. When in motion:

$$\underbrace{(\,)}_{\Delta x} \;\lambda \longrightarrow v \qquad \Delta x = \frac{1}{\gamma}\Delta x_0 \text{ (length contraction)}$$

the charge in the interval is still q, but now $q = \lambda \Delta x = \lambda \frac{1}{\gamma}\Delta x_0$. Equating the charge in the interval:

$$\lambda_0 \Delta x_0 = \lambda \frac{1}{\gamma}\Delta x_0 \Rightarrow \lambda = \gamma \lambda_0 > \lambda_0$$

For a dipole at rest in $\bar{L}$, that moves through L w/ speed v:



$$\bar{I} = 2\bar{\lambda}\bar{v} \quad \text{w/}$$

$$\bar{\lambda} = \gamma_{\bar{v}} \lambda_0 \qquad \left(\gamma_{\bar{v}} = \frac{1}{\sqrt{1 - \bar{v}^2/c^2}} \right)$$

we have neutral current in $\bar{L}$, w/ $\quad \begin{array}{c} \bar{\lambda} \\ \hline \end{array} \to \bar{v} \quad \bar{v} \leftarrow \begin{array}{c} -\bar{\lambda} \\ \hline \end{array}$ as our model

The total charge along the bottom leg of the circuit is $\bar{Q}_b = \bar{\lambda}\bar{a} - \bar{\lambda}\bar{a} = 0$

The magnetic dipole moment in $\bar{L}$ is $\bar{m} = \bar{I}\bar{a}^2 = 2\gamma_{\bar{v}} \lambda_0 \bar{v} \bar{a}^2$

In L, the bottom leg of the circuit looks like:

L:

$$\begin{array}{c} \lambda_1 = \gamma_u \lambda_0 \\ \hline \end{array} \longrightarrow u = \frac{\bar{v} + v}{1 + v\bar{v}/c^2}$$

$$w = \frac{v - \bar{v}}{1 - v\bar{v}/c^2} \longleftarrow \begin{array}{c} \lambda_2 = \gamma_w \lambda_0 \\ \hline \end{array}$$

$\Big\}$ total charge $\neq 0$ since $\gamma_u \neq \gamma_w$

In addition to the γ_u, γ_w mismatch, there are the side lengths to consider – for the bottom leg,

$$a = \frac{1}{\gamma_v}\bar{a} \quad \text{(length contraction)}$$

this also happens for the top segment. The sides remain at $\bar{a}$ ($\perp$ to boost)

The total charge on the bottom leg is:

$$Q_b = (\gamma_u - \gamma_w)\lambda_0 a = \frac{\gamma_u - \gamma_w}{\gamma_v}\lambda_0 \bar{a}$$

& $u > w$, so $\gamma_u > \gamma_w$, so $Q_b < 0$

The top leg will have $Q_t = -Q_b$, so the dipole moment will be:

$$\vec{p} = |Q_b|\bar{a}(-\hat{y})$$

we have the relations:

$$\gamma_u = \gamma_v \gamma_{\bar{v}}(1 + v\bar{v}/c^2)$$

$$\gamma_w = \gamma_v \gamma_{\bar{v}}(1 - v\bar{v}/c^2)$$

$$\frac{\gamma_u - \gamma_w}{\gamma_v} = 2\gamma_{\bar{v}}\frac{v\bar{v}}{c^2}$$

the magnitude of the dipole moment is:

$$p = (2\gamma_{\bar{v}}\lambda_0 \bar{v}\bar{a}^2)v/c^2 = \bar{m}\, v/c^2$$

the direction of $\vec{\bar{m}}$ is $\hat{z}$, & for $\vec{p} \sim -\hat{y}$, so

$$\boxed{\vec{p} = \frac{\vec{v} \times \vec{\bar{m}}}{c^2}}$$

Electron at Rest

the proton moves in a circle around the e^-, acting as a smeared out charge w/

$$\lambda = \frac{e}{2\pi r} \quad \& \quad I = \lambda v = \frac{e}{T}$$

at the center of a steady loop of current: $B = \frac{\mu_0 I}{2r} = \frac{\mu_0 e}{2rT}$

& it points upward, let's express $\vec{B}$ in terms of the orbital angular momentum of the e^- – the Bohr story is:

$$\vec{L} = \vec{r} \times \vec{p} = r m_e v\,\hat{z} = r m_e\left(\frac{2\pi r}{T}\right)\hat{z}$$

$$= \frac{2\pi m_e r^2}{T}\hat{z} \Rightarrow T = \frac{2\pi m_e r^2}{L}$$

the magnetic field can be written as: $\vec{B} = \frac{\mu_0 e}{2r}\left(\frac{\vec{L}}{2\pi m_e r^2}\right)$

or $\vec{B} = \frac{\mu_0 e}{4\pi m_e r^3}\vec{L}$

We can estimate the magnetic dipole moment of the e^-:

the effective current is $I = -e/T$, so

$m = e/T\, \pi a^2$ is the magnitude.

the angular momentum of the ring of e^- mass m_e is:

$$S = I\omega = \frac{m_e \cdot 2\pi a^2}{T} \quad \& \quad \frac{m}{S} = \frac{e}{2m_e} \Rightarrow \vec{m} = \frac{-e}{2m_e}\vec{S}$$

It turns out that the electron's magnetic dipole moment is 2x this (relativistic correction):

$$\vec m = -\frac{e}{m_e}\vec S$$

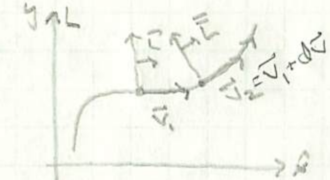
Now the perturbation "potential" is

$$U' = -\vec m\cdot\vec B = \frac{\mu_0 e^2}{4\pi m_e^2 r^3}\,\vec S\cdot\vec L \longrightarrow$$

Issue: we have analyzed the problem in the e^- rest frame - but the e^- is accelerating

As we follow the e^- on its journey, we need to apply different boosts into its rest frame at different points.

general cartoon:



the relation between the successive $\bar L$ & $\bar{\bar L}$ frames is given by:

$$\bar x^\mu = \underbrace{\Lambda^\mu{}_\nu(\vec v_1)}_{\text{boost matrix along } \vec v_1 \text{ w/ speed } v_1} x^\nu \qquad \bar{\bar x}^\mu = \Lambda^\mu{}_\nu(\vec v_1 + d\vec v)\, x^\nu$$

$$x^\nu = \Lambda^\nu{}_\rho(-\vec v_1)\,\bar x^\rho$$

so that $\bar{\bar x}^\mu$ relative to $\bar x^\mu$ is:

$$\bar{\bar x}^\mu = \Lambda^\mu{}_\nu(\vec v_1 + d\vec v)\, x^\nu = \Lambda^\mu{}_\nu(\vec v_1 + d\vec v)\,\Lambda^\nu{}_\rho(-\vec v_1)\,\bar x^\rho$$

$\bar{\bar x}^\mu$ is related to $\bar x^\mu$ by 2 boosts in different directions.

We know that successive boosts in different directions can induce a rotation (the "classical" commutators mix together boosts w/ rotations), & that changes the rotat - effect is called Thomas precession - ends up halving our predicted interaction:

$$U' = \frac{\mu_0 e^2}{8\pi m_e^2 r^3}\,\vec S\cdot\vec L$$