Degeneracy

A mass is constrained to a ring of radius R:

$$-\frac{\hbar^2}{2mR^2}\frac{d^2\psi(\phi)}{d\phi^2} = E\psi(\phi) \quad (*)$$

w/ $\psi(\phi+2\pi) = \psi(\phi)$.

One solution is $\psi_+ = e^{ipR\phi/\hbar}$ w/ $p = \sqrt{2mE}$

This is an eigenstate of $\hat{H} = \hat{p}^2/2m$ (free particle on the ring) and the "$\hat{T}$" operator (for angles, $\phi \to \phi - \varphi$).

$\hat{H}\psi_+ = E\psi_+$ & since $[\hat{H},\hat{T}] = 0$,

$\hat{H}(\hat{T}\psi_+) = E(\hat{T}\psi_+)$ so $\hat{T}\psi_+$ also has energy E.

But $\hat{T}\psi_+ = e^{-ipR\varphi/\hbar}\cdot\psi_+ \sim \psi_+$, the same state

The parity operator (for angles, $\phi \to -\phi$) commutes w/ $\hat{H}$, so

$\hat{H}(\hat{\Pi}\psi_+) = E(\hat{\Pi}\psi_+)$ & $\hat{\Pi}\psi_+$ is a state w/ energy E,
but
$$\hat{\Pi}\psi_+ = e^{-ipR\phi/\hbar}$$

this is a new state, $\hat{\Pi}\psi_+ \not\propto \psi_+$, i.e. ψ_+ is not an eigenstate of $\hat{\Pi}$, not a surprise since $[\hat{\Pi},\hat{T}] \neq 0$!

Degeneracy happens when $[\hat{H},\hat{A}] = 0$, $[\hat{H},\hat{B}] = 0$, $[\hat{A},\hat{B}] \neq 0$.

? This example is $D=1$, don't we have a theorem about that?

Perturbation Theory Example

Let's find the 1st order corrections,

$$E_j^1 = \langle\psi_j^0|\hat{H}'|\psi_j^0\rangle, \quad |\psi_j^1\rangle = \sum_{\substack{k=0\\ \neq j}}^{\infty} \frac{\langle\psi_k^0|\hat{H}'|\psi_j^0\rangle}{E_j^0 - E_k^0}|\psi_k^0\rangle$$

for a simple case. For the "zero" problem, we'll use the infinite sq. well (width a):

$$\hat{H}^0 = \hat{p}^2/2m \quad \text{w/} \quad \psi_j^0(x) = \sqrt{\tfrac{2}{a}}\sin(j\pi x/a), \quad E_j^0 = \frac{j^2\pi^2\hbar^2}{2ma^2}$$

& take $\hat{H}' = u\,\delta(x - a/2)$ for small u.

$$E_j^1 = \langle\psi_j^0|\hat{H}'|\psi_j^0\rangle = \int_{-\infty}^{+\infty}\psi_j^*(x)\,u\,\delta(x-a/2)\,\psi_j(x)\,dx \overset{!}{=} \frac{2u}{a}\sin^2(j\pi/2)$$

so $E_j \approx E_j^0 + E_j^1 = \dfrac{j^2\pi^2\hbar^2}{2ma^2} + \dfrac{2u}{a}\sin^2\left(\dfrac{j\pi}{2}\right)$

note that only the odd (in j) states pick up corrections ...?

To get $|\psi_j^1\rangle$, we need $\langle\psi_k^0|\hat{H}'|\psi_j^0\rangle$:

$$\langle\psi_k^0|\hat{H}'|\psi_j^0\rangle = \frac{2}{a}u\sin\left(\frac{k\pi}{2}\right)\sin\left(\frac{j\pi}{2}\right) = \begin{cases}\frac{2u}{a} & \text{if } j,k \text{ odd}\\ 0 & \text{else}\end{cases}$$

$$\psi_j^1(x) = \sum_{\substack{k=1,3,5\\ \neq j}}^{\infty}\left(\frac{2u/a}{\frac{(j^2-k^2)\pi^2\hbar^2}{2ma^2}}\right)\sqrt{\frac{2}{a}}\sin\left(\frac{k\pi x}{a}\right)$$

Suppose we use the same perturbation, but started from the mass on a ring:

$u\delta(\phi)$

$$\hat{H}^0 = \hat{p}^2/2m \qquad \psi_j^0 = A e^{ij\phi} \qquad E_j^0 = \frac{j^2\hbar^2}{2mR^2}$$

norm

w/ $j = -\infty \to \infty$.

The corrections to the states look like:

$$\psi_j^1(\phi) = \sum_{\substack{k=-\infty \\ k\neq j}}^{+\infty} \frac{\langle \psi_k^0 | \hat{H}' | \psi_j^0 \rangle}{E_j^0 - E_k^0} \psi_k^0(\phi)$$

& $E_j^0 - E_k^0 = 0$ if $k = -j$, the degeneracy has introduced $1/0$ ...

What should we do? Before we get started, note that while:

$[\hat{H}^0, \hat{\Pi}] = 0$ & $[\hat{H}^0, \hat{T}] = 0$ here, the commutators w/ the full Hamiltonian are:

$$[\hat{H}^0 + \hat{H}', \hat{\Pi}] = 0, \text{ but } [\hat{H}^0 + \hat{H}', \hat{T}] = 0$$

we do not expect degeneracy in the spectrum of $\hat{H}^0 + \hat{H}'$.

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There is an issue prior to $|\psi_j^1\rangle$, though - any linear combo. of ψ_j^0 & ψ_{-j}^0 also has energy E_j^0 - how can we find E_j^1 if we don't know which combo to use in

$$\langle \psi_j^0 | \hat{H}' | \psi_j^0 \rangle \; ?$$

Degenerate Perturbation Theory

Model setup - we have $\hat{H}^0$ w/ 2 states $|\psi_a^0\rangle$ & $|\psi_b^0\rangle$ that both have energy E^0:

$$\hat{H}^0 |\psi_a^0\rangle = E^0 |\psi_a^0\rangle \qquad \hat{H}^0 |\psi_b^0\rangle = E^0 |\psi_b^0\rangle$$

and

$$\langle \psi_a^0 | \psi_b^0 \rangle = 0.$$

the idea is to pick a linear combination

$$|\psi^0\rangle \equiv \alpha |\psi_a^0\rangle + \beta |\psi_b^0\rangle$$

then the perturbation problem is to find $|\psi^1\rangle$ in the 1st order eqn.:

$$(\hat{H}^0 + \lambda \hat{H}')(|\psi^0\rangle + \lambda |\psi^1\rangle) = (E^0 + \lambda E^1)(|\psi^0\rangle + \lambda |\psi^1\rangle)$$

$$\text{w}/\ \lambda^1: \ \hat{H}'|\psi^0\rangle + \hat{H}^0|\psi^1\rangle = E^0|\psi^1\rangle + E^1|\psi^0\rangle$$

hit both sides of this eqn. w/ $\langle \psi_a^0 |$

$$\langle \psi_a^0 | \hat{H}' | \psi^0 \rangle + \underbrace{\langle \psi_a^0 | \hat{H}^0 | \psi^1 \rangle}_{= E^0 \langle \psi_a^0 | \psi^1 \rangle \leftarrow \text{cancel}} = E^0 \langle \psi_a^0 | \psi^1 \rangle + E^1 \underbrace{\langle \psi_a^0 | \psi^0 \rangle}_{=\alpha}$$

$$\alpha \langle \psi_a^0 | \hat{H}' | \psi_a^0 \rangle + \beta \langle \psi_a^0 | \hat{H}' | \psi_b^0 \rangle = \alpha E^1 \qquad (*)$$

& hitting that eqn. w/ $\langle \psi_b^0 |$ gives

$$\alpha \langle \psi_b^0 | \hat{H}' | \psi_a^0 \rangle + \beta \langle \psi_b^0 | \hat{H}' | \psi_b^0 \rangle = \beta E^1 \qquad (o)$$

Let $W_{xy} \equiv \langle \psi_x^0 | \hat{H}' | \psi_y^0 \rangle$, then the pair $(*)$ & (o) can be written:

$$\underbrace{\begin{pmatrix} W_{aa} & W_{ab} \\ W_{ba} & W_{bb} \end{pmatrix}}_{\equiv \mathbb{W} = \mathbb{W}^\dagger} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = E^1 \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \text{ an \_\_\_\_\_ problem}$$

There will be 2 solutions, the eigenvalues/vecs of the matrix $\mathbb{W}$, call them $\begin{pmatrix}\alpha_\pm \\ \beta_\pm\end{pmatrix}$ associated w/ $E_\pm$.
→ let:

$$|\psi_+^0\rangle = \alpha_+|\psi_a^0\rangle + \beta_+|\psi_b^0\rangle$$
$$|\psi_-^0\rangle = \alpha_-|\psi_a^0\rangle + \beta_-|\psi_b^0\rangle$$
w/ $\langle\psi_+^0|\psi_-^0\rangle = 0$ ($\mathbb{W}^\dagger = \mathbb{W}$)

we now know the "correct" linear combination to use in the energy correction; & the value of the correction.

If you started w/ $|\psi_+^0\rangle$ & $|\psi_-^0\rangle$,

$\langle\psi_+^0|\hat{H}'|\psi_+^0\rangle = E_+^1$ & $\langle\psi_-^0|\hat{H}'|\psi_-^0\rangle = E_-^1$

w/ $E \approx E_0 + E_+^1$ w/ $E \approx E_0 + E_-^1$

If $E_+^1 \neq E_-^1$, the degeneracy has been "lifted."

Fine - but we had to do a lot of work to find this linear combination.

There has to be a better way...

"Good State Picking"

The goal is combinations of states w/ the same energy for which
$$\mathbb{W} = \begin{pmatrix} W_{aa} & 0 \\ 0 & W_{bb}\end{pmatrix}$$, $\mathbb{W}$ is diagonal.

that means $\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \begin{pmatrix}1 \\ 0\end{pmatrix}$ & $\begin{pmatrix}0 \\ 1\end{pmatrix}$, so we've chosen the eigenvectors of $\mathbb{W}$ as the initial linear combinations.

Algorithm: Find Hermitian $\hat{A}$ w/ $[\hat{H}^0, \hat{A}] = 0$ & $[\hat{H}', \hat{A}] = 0$
& has
$$\hat{A}|\psi_a^0\rangle = \mu|\psi_a^0\rangle \qquad \hat{A}|\psi_b^0\rangle = \nu|\psi_b^0\rangle$$
w/ $\mu \neq \nu$ (simultaneous eigenstates of $\hat{H}$ & $\hat{A}$ are available here, since $[\hat{H}, \hat{A}] = 0$).

then: $\langle\psi_a^0|[\hat{H}', \hat{A}]|\psi_b^0\rangle = 0$

$$\langle\psi_a^0|\hat{H}'\hat{A}|\psi_b\rangle = \langle\psi_a^0|\hat{A}\hat{H}'|\psi_b^0\rangle$$
$$\nu\langle\psi_a^0|\hat{H}'|\psi_b^0\rangle = \mu\langle\psi_a^0|\hat{H}'|\psi_b^0\rangle$$

→ $\mu \neq \nu$, so $\langle\psi_a^0|\hat{H}'|\psi_b^0\rangle = 0$, but this is W_{ab}.
→ sim. $\langle\psi_b^0|\hat{H}'|\psi_a^0\rangle = W_{ba} = 0$.

For our mass on a ring, we have: $[\hat{H}^0, \hat{T}] = 0$, $[\hat{H}^0, \hat{\Pi}] = 0$ but only $\hat{\Pi}$ commutes with the perturbation, so we should start off w/ the simultaneous eigenstates of $\hat{H}$ & $\hat{\Pi}$:

$$\psi_j(x) = \bar{A}\cos(j\phi) \quad \& \quad \bar{B}\sin(j\phi).$$