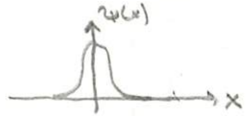
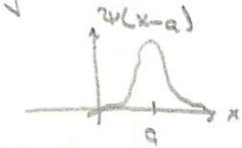
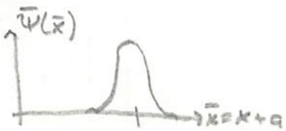
Last Time



shift function to the right,



or shift axis to the left



For $\psi(x-a)$, we saw:

$$\psi(x-a) = \underbrace{\left(\sum_{j=0}^{\infty} \frac{(-a)^j}{j!} \frac{d^j}{dx^j} \right) \psi(x)} = e^{-a\frac{d}{dx}} \psi(x)$$

the spatial translation operator is:

$$\hat{T}(a) = e^{-i/\hbar\, a \hat{p}}$$

$\left(\hat{p} = \frac{\hbar}{i}\frac{d}{dx}\right)$ w/

$\hat{T}(a)\psi(x) = \psi(x-a)$.

Properties of $\hat{T}$

$$\hat{T}(a) = e^{-i/\hbar\, a \hat{p}}$$

infinitesimal form:

$$\hat{T}(\epsilon) \approx \left(1 - \tfrac{i}{\hbar}\epsilon \hat{p}\right)$$

($\hat{p}$ is the "generator" of translation)

Inverse: $\hat{T}(-a)\psi(x) = \psi(x+a)$, so

$$\hat{T}(-a) = \hat{T}^{-1}(a)$$

$$\hat{T}(-a)\hat{T}(a)f(x) = \hat{T}(-a)f(x-a) \overset{!}{=} f(x) \checkmark$$

$$\hat{T}(-a) = e^{i/\hbar\, a \hat{p}} = \hat{T}(a)^\dagger$$

so $\hat{T}^{-1} = \hat{T}^\dagger$ & $\hat{T}$ is unitary.

we're working in the position basis.

$\hat{T}(a)\psi(x) = \psi(x-a)$ is really:

$\langle x|\hat{T}|\psi\rangle$.

w/ $\hat{T}|\psi\rangle$ the "abstract" ket form.

For $|\bar{\psi}\rangle = \hat{T}|\psi\rangle$, we have

$$\langle x|\bar{\psi}\rangle = \langle x|\hat{T}|\psi\rangle$$

$$\bar{\psi}(x) = \psi(x-a) \checkmark$$

Operator Transformation

We can also define the response of operators to $\hat{T}$.

For $\hat{Q}(\hat{x},\hat{p})$, w/ $|\bar{\psi}\rangle = \hat{T}|\psi\rangle$, we can compute the expectation value of $\hat{Q}$:

$$\langle\bar{\psi}|\hat{Q}|\bar{\psi}\rangle = \langle\psi|\hat{T}^\dagger\hat{Q}\hat{T}|\psi\rangle \equiv \langle\psi|\bar{\hat{Q}}|\psi\rangle$$

so that $\bar{\hat{Q}} \equiv \hat{T}^\dagger\hat{Q}\hat{T}$

example: $\hat{x}$ acts on $f(x)$ via: $\hat{x}f(x) = xf(x)$.
? what is $\bar{\hat{x}} = $?

$$\bar{\hat{x}}f(x) = \hat{T}^\dagger\hat{x}\hat{T}f(x) = \hat{T}^\dagger\hat{x}f(x-a) = \hat{T}^\dagger \underbrace{x f(x-a)}_{\equiv g(x)} = \hat{T}^\dagger g(x) = g(x+a) = (x+a)f(x)$$

then $\bar{\hat{x}} = \hat{x} + a$ (shift coordinates left ✓).

example: $\hat{p}f(x) = \frac{\hbar}{i}\frac{df}{dx}$ what is $\bar{\hat{p}} = $?

$$\bar{\hat{p}}f(x) = \hat{T}^\dagger\hat{p}\hat{T}f(x) = \hat{T}^\dagger\frac{\hbar}{i}\frac{df(x-a)}{dx} = \frac{\hbar}{i}\frac{df(x)}{dx} \Rightarrow \bar{\hat{p}} = \hat{p}$$

(no surprise here: $d\bar{x} = d(x+a) = dx$ ✓)

note that $\bar{\hat{p}} = \hat{p} \Rightarrow \hat{T}^\dagger\hat{p}\hat{T} = \hat{p}$

$$\hat{p}\hat{T} - \hat{T}\hat{p} = [\hat{T},\hat{p}] = 0.$$

For a generic operator $\hat{Q}(\hat{x},\hat{p})$, we can expand in the $\hat{x}$, $\hat{p}$ operators:

$$\hat{Q} = \sum_{ij} c_{ij}\, \hat{x}^i \hat{p}^j \quad \text{w/}$$

$$\hat{\bar{Q}} = \hat{T}^\dagger \hat{Q} \hat{T} = \sum_{ij} c_{ij}\, \hat{T}^\dagger \hat{x}^i \underbrace{\hat{p}^j \hat{T}}_{=\hat{T}\hat{p}^j}$$

$$= \sum_{ij} c_{ij} \big(\underbrace{\hat{T}^\dagger \hat{x} \hat{T}}_{\hat{\bar{x}} = \hat{x}+a}\big)^i \hat{p}^j$$

$$= \sum_{ij} c_{ij} (\hat{x}+a)^i \hat{p}^j$$

or $\hat{\bar{Q}}(\hat{\bar{x}},\hat{\bar{p}}) = \hat{Q}(\hat{x}+a,\hat{p})$

Noether's Theorem

How does $\hat{H}$ respond to coordinate translation?

$$\hat{\bar{H}} = \hat{T}^\dagger \hat{H} \hat{T} = \hat{T}(-\epsilon)\, \hat{H}\, \hat{T}(\epsilon)$$ (working infinitesimally, as we did in CM)

$$\approx (1 + i\epsilon/\hbar\, \hat{p})\, \hat{H}\, (1 - i\epsilon/\hbar\, \hat{p})$$

$$= \hat{H} + i/\hbar\, \epsilon [\hat{p},\hat{H}] + O(\epsilon^2)$$

and we also have Ehrenfest's theorem

$$\frac{d\langle \hat{p}\rangle}{dt} = \frac{i}{\hbar}\langle [\hat{H},\hat{p}]\rangle$$

so that if $\hat{\bar{H}} = \hat{H}$ (Hamiltonian operator is unchanged by coord. translation), then

$$[\hat{p},\hat{H}] = 0 \;\rightarrow\; \frac{d\langle \hat{p}\rangle}{dt} = 0$$

the expectation value of momentum is conserved.

(CM: $\bar{H}(\bar{x},\bar{p}) = H(x,p) + \{H,J\}_{P.B.}$

$\frac{dJ}{dt} = \{J,H\}_{P.B.}$ if $\bar{H}(\bar{x},\bar{p}) = H(x,p)$, $\{H,J\}_{PB}=0$ → then $\frac{dJ}{dt} = 0$ ✓)

In CM, the question of symmetry ⟹ conservation relies on a transformation generated by J that is continuous (derivs of J)!

In QM, symmetry ⟹ conservation is more robust

$[\hat{H},\hat{Q}] = 0$ can happen even if $\hat{Q}$ does not generate a continuous transformation.

Conservation in Quantum Mechanics

As stated above, Noether's theorem has, as its QM "conservation"

$$\frac{d\langle \hat{Q}\rangle}{dt} = 0$$

the expectation value of the operator $\hat{Q}$ is t-independent.

An alternate view: for $\hat{Q}$ w/ $\hat{Q}|f_n\rangle = q_n|f_n\rangle$ possible measurements are $\{q_n\}_{n=0}^{\infty}$

let $P(q_n,t)$ = prob. of getting q_n upon $\hat{Q}$ measurement.
as a function of time.

By QM conservation, you might mean:

$$\frac{dP(q_n,t)}{dt} = 0 \qquad (*)$$

The two versions, $\frac{d\langle\hat{Q}\rangle}{dt}=0$, & $(*)$ are equivalent.

If $\frac{d\langle\hat{Q}\rangle}{dt}=0$, then $[\hat{H},\hat{Q}]=0$, from

$$\frac{d\langle\hat{Q}\rangle}{dt} = \frac{i}{\hbar}\langle[\hat{H},\hat{Q}]\rangle$$

For the 2nd $(*)$ form, if you have:

$$|\psi(t)\rangle = \sum_{j=0}^{\infty} c_j e^{-iE_j t/\hbar}|\psi_j\rangle$$

w/ $\hat{H}|\psi_j\rangle = E_j|\psi_j\rangle$, then

$$P(q_n,t) = |\langle f_n|\psi(t)\rangle|^2 = \left|\sum_{j=0}^{\infty} c_j e^{-iE_j t/\hbar}\langle f_n|\psi_j\rangle\right|^2$$

the sum involves the bases of the $\hat{Q}$ & $\hat{H}$ operators. If $[\hat{H},\hat{Q}]=0$, then $\exists$ a basis in which

$$\langle f_n|\psi_j\rangle = \delta_{nj} \quad \text{(simultaneous diagonalizability)}$$

then $P(q_n,t) = |c_n|^2$ ← a constant.

$$\frac{dP(q_n,t)}{dt} = 0 \checkmark$$

Translational Symmetry

For our $\hat{T}$ operator, we know that the relevant commutator is

for Noether's thm.

$$[\hat{H},\hat{p}] = \hat{H}\hat{p} - \hat{p}\hat{H}$$

& for $\hat{H} = \hat{p}^2/2m + U(\hat{x})$,

$$[\hat{H},\hat{p}] = [U(\hat{x}),\hat{p}]$$

use a test-function to compute:

$$[U(\hat{x}),\hat{p}]f(x) = U(x)\frac{\hbar}{i}\frac{df}{dx} - \frac{\hbar}{i}\frac{d}{dx}(U(x)\cdot f(x))$$

$$= \left(-\frac{\hbar}{i}\frac{dU}{dx}\right)f(x)$$

so $[U(\hat{x}),\hat{p}] = -\frac{\hbar}{i}\frac{dU}{dx}$

& in order to get zero, we have the (CM) restriction

$$\frac{dU}{dx}=0 \qquad \text{no potential energy}$$

There is another option, though:

$$[\hat{H},\hat{T}(a)] = \hat{H}\hat{T}(a) - \hat{T}(a)\hat{H} = 0 = [U(\hat{x}),\hat{T}(a)]$$

is broader than $[\hat{H},\hat{p}]=0$.

$$[U(\hat{x}),\hat{T}(a)]f(x) = U(x)f(x-a) - U(x-a)f(x-a)$$

If $U(x) = U(x-a)$, then $[U(\hat{x}),\hat{T}(a)] = 0$ ✓

periodic potential ...