Transformation in Classical Mechanics

$L = \frac{1}{2}mv^2 - U(x)$ has the advantage of coord.-independence.

For $H(x,p)$, the coordinate choice & momenta are linked.

example: $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2$

w/ eqn. of motion: $-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{\partial L}{\partial x} = 0$

$$\Downarrow$$

$$m\ddot{x} = -m\omega^2 x$$

let $\bar{x} = \alpha x$, $\bar{L} = \frac{1}{2}m\,\dot{\bar{x}}^2/\alpha^2 - \frac{1}{2}m\omega^2\bar{x}^2/\alpha^2$

w/

$$-\frac{d}{dt}\frac{\partial \bar{L}}{\partial \dot{\bar{x}}} + \frac{\partial \bar{L}}{\partial \bar{x}} = 0 \Rightarrow m\ddot{\bar{x}} = -m\omega^2\bar{x} \checkmark$$

the analogous story for $H = p^2/2m + \frac{1}{2}m\omega^2 x^2$ is

eqns of motion: $\dot{x} = \frac{\partial H}{\partial p} = p/m \Rightarrow p = m\dot{x}$

$$\dot{p} = -\frac{\partial H}{\partial x} = -m\omega^2 x \Rightarrow m\ddot{x} = -m\omega^2 x \checkmark$$

but for $\bar{x} = \alpha x$, $\bar{H} = p^2/2m + \frac{1}{2}m\omega^2\bar{x}^2/\alpha^2$

$$\dot{\bar{x}} = \frac{\partial H}{\partial p} = p/m \Rightarrow p = m\dot{\bar{x}}$$

$$\dot{p} = -\frac{\partial H}{\partial \bar{x}} = -m\omega^2\bar{x}/\alpha^2 \text{ leading to}$$

$$m\ddot{\bar{x}} = -m(\omega/\alpha)^2\bar{x} \quad \ddot{\frown}$$

Generating Functions to ensure the correct link between $\dot{x}$ & p, take: $K(x,\bar{p})$, then

$$\bar{x} = \frac{\partial K}{\partial \bar{p}} \qquad p = \frac{\partial K}{\partial x}$$

ensures that $\bar{H}(\bar{x},\bar{p})$ has the same eqn.s of motion as $H(x,p)$.

example: for $\bar{x} = \alpha x$, $K = \alpha\bar{p}x$, then $p = \frac{\partial K}{\partial x} = \alpha\bar{p}$
so $\bar{p} = p/\alpha$.

$+\ \bar{H}(\bar{x},\bar{p}) = \alpha^2\bar{p}^2/2m + \frac{1}{2}m\omega^2\bar{x}^2/\alpha^2$

w/ $\dot{\bar{x}} = \frac{\partial H}{\partial \bar{p}} = \alpha^2\bar{p}/m \Rightarrow m\dot{\bar{x}} = \alpha^2\bar{p}$

$$\dot{\bar{p}} = -\frac{\partial H}{\partial \bar{x}} = -m\omega^2\bar{x}/\alpha^2 \Rightarrow m\ddot{\bar{x}} = -m\omega^2\bar{x}^2 \checkmark$$

Infinitesimal Generating Functions

The identity is generated by $K = x\bar{p}$: $\bar{x} = \frac{\partial K}{\partial \bar{p}} = x$, $p = \frac{\partial K}{\partial x} = \bar{p} \checkmark$

Take $K(x,\bar{p}) = x\bar{p} + \epsilon J(x,\bar{p})$, then:

$$\bar{x} = \frac{\partial K}{\partial \bar{p}} = x + \epsilon\frac{\partial J}{\partial \bar{p}} \qquad p = \frac{\partial K}{\partial x} = \bar{p} + \epsilon\frac{\partial J}{\partial x} \Rightarrow \bar{p} = p - \epsilon\frac{\partial J}{\partial x}$$

$$= x + \epsilon\frac{\partial J}{\partial p}\cdot\underbrace{\frac{\partial \bar{p}}{\partial p}}_{=O(\epsilon)} = x + \epsilon\frac{\partial J}{\partial p}.$$

so take $J(x,p)$, the infinitesimal "canonical" transformation is:

$$\bar{x} = x + \epsilon\frac{\partial J}{\partial p} \qquad \bar{p} = p - \epsilon\frac{\partial J}{\partial x}$$

Noether's Theorem

If the transformation generated by $J(x,p)$ leaves the Hamiltonian unchanged (form invariant), then $J(x,p)$ is a constant of motion.

we've seen: $\frac{dJ}{dt} = \frac{\partial J}{\partial x}\dot{x} + \frac{\partial J}{\partial p}\dot{p} = \frac{\partial J}{\partial x}\frac{\partial H}{\partial p} - \frac{\partial J}{\partial p}\frac{\partial H}{\partial x}$

$= \{J, H\}_{P.B.}$ (*)

How does H respond to the coord. transformation generated by J?

$\bar{H}(\bar{x},\bar{p}) = H\left(x + \epsilon\frac{\partial J}{\partial p}, p - \epsilon\frac{\partial J}{\partial x}\right)$

$\approx H(x,p) + \epsilon\underbrace{\left(\frac{\partial H}{\partial x}\frac{\partial J}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial J}{\partial x}\right)}_{=\{H,J\}}$

$\to \bar{H}(\bar{x},\bar{p}) = H(x,p)$ if $\{H,J\} = 0$

symmetry

but then (*) says $\frac{dJ}{dt} = -\{H,J\}_{PB} = 0$

conservation.

example: $J = H$ has $\{H,H\}_{P.B.} = 0$ automatically.

what transformation is generated by H?

$\bar{x} = x + \epsilon\frac{\partial H}{\partial p} = x + \epsilon\dot{x}$

$\frac{\partial H}{\partial p} = \dot{x}$

$\bar{p} = p - \epsilon\frac{\partial H}{\partial x} = p + \epsilon\dot{p}$

$\frac{\partial H}{\partial x} = -\dot{p}$

write these w/ t in place: $\bar{x}(t) = x(t) + \epsilon\dot{x}(t) \approx x(t+\epsilon)$

$\bar{p}(t) = p(t) + \epsilon\dot{p}(t) \approx p(t+\epsilon)$.

time-translation invariance.

example: how about $J = p$ - what transformation is "generated by momentum"?

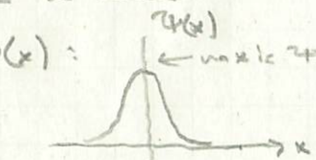
$\bar{x} = x + \epsilon \qquad p = \bar{p}$

spatial translation. And p is conserved if $\{J,H\}_{P.B.} = 0$

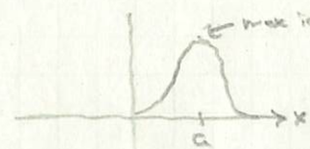
(? what does this tell us?)

Coordinate Translation in QM

Given a function $\psi(x)$:

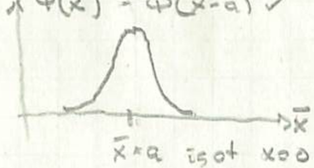


we can plot $\psi(x-a)$: a shift to the right.



we could, instead, shift the coordinates to the left: $\bar{x} = x + a$

$\bar{\psi}(\bar{x}) = \psi(\bar{x} - a)$ ✓



Taking the first view, what is the relation between $\psi(x-a)$ and $\psi(x)$?

$$\psi(x-a) = \sum_{j=0}^{\infty} \frac{1}{j!}(-a)^j \frac{d^j\psi}{dx^j} = \left[\sum_{j=0}^{\infty} \frac{1}{j!}(-a)^j \frac{d^j}{dx^j}\right]\psi(x)$$

$$= e^{-a\frac{d}{dx}}\psi(x)$$

and in QM, $\hat{p} = \frac{\hbar}{i}\frac{d}{dx}$, so

$$\psi(x-a) = e^{-ia\hat{p}/\hbar}\psi(x)$$

define the operator $\hat{T}(a) \equiv e^{-ia\hat{p}/\hbar}$, w/ infinitesimal form:

$$\hat{T}(\epsilon) \approx 1 - \frac{i\epsilon}{\hbar}\hat{p}$$

momentum is, again, the generator of translation.

The inverse of $\hat{T}(\epsilon)$ is $\hat{T}(-\epsilon)$:

$$\hat{T}(-\epsilon)\hat{T}(\epsilon)\psi(x) = \hat{T}(-\epsilon)\psi(x-\epsilon) \overset{!}{=} \psi(x)$$

so $\hat{T}(\epsilon)^{-1} = \hat{T}(-\epsilon) = e^{i\epsilon\hat{p}/\hbar} = \hat{T}(\epsilon)^\dagger$

& the operator $\hat{T}(a)$ is "unitary": $\hat{T}^{-1} = \hat{T}^\dagger$

Operator Transformation

We've defined the response of the wavefunction to $\hat{T}$, but what about the response of other operators?

Given $\hat{Q}(\hat{x},\hat{p})$, what do we mean by $\hat{\bar{Q}}$?

For $|\bar{\psi}\rangle \equiv \hat{T}|\psi\rangle$ (w/ $\langle x|\bar{\psi}\rangle = \langle x|\hat{T}\psi\rangle$; $\bar{\psi}(x) = \psi(x-a)$)

the expectation value of $\hat{Q}$ is:

$$\langle\bar{\psi}|\hat{Q}|\bar{\psi}\rangle = \langle\psi|\hat{T}^\dagger\hat{Q}\hat{T}|\psi\rangle \equiv \langle\psi|\hat{\bar{Q}}|\psi\rangle$$

so $\hat{\bar{Q}} = \hat{T}^\dagger\hat{Q}\hat{T}$

example for $\hat{x}f(x) = xf(x)$ (position operator), what is

$$\hat{\bar{x}} = \hat{T}^\dagger\hat{x}\hat{T} \quad ?$$

$$\hat{\bar{x}}f(x) = \hat{T}^\dagger\hat{x}\hat{T}f(x) = \hat{T}^\dagger\hat{x}f(x-a) \overset{!}{=} \hat{T}^\dagger\underbrace{x f(x-a)}_{\equiv g(x)} \overset{!}{=} g(x+a) \overset{!}{=} (x+a)f(x)$$

so $\hat{\bar{x}} = x + a$ (shifting coordinates)