Heisenberg Operators

In the Schrödinger Picture, states move in time, operators do not: $\hat{Q}_s|\Psi(t)\rangle$, w/

$$\langle \hat{Q}_s\rangle(t) = \langle\Psi(t)|\hat{Q}_s|\Psi(t)\rangle$$

In the Heisenberg Picture, operators move, & states do not: $\hat{Q}_H(t)|\Psi(0)\rangle$. w/

$$\langle \hat{Q}_H\rangle(t) = \langle\Psi(0)|\hat{Q}_H(t)|\Psi(0)\rangle$$

Heisenberg operators are defined by:

$$\hat{Q}_H = \hat{U}^+(t)\,\hat{Q}_s\,\hat{U}(t)$$

time-translation op. $\hat{U} = e^{-i/\hbar\, t\hat{H}}$

so that:

$$\begin{aligned}\langle\hat{Q}_H\rangle(t) &= \langle\Psi(0)|\hat{Q}_H(t)|\Psi(0)\rangle \\ &= \langle\Psi(0)|\hat{U}^+\hat{Q}_s\hat{U}|\Psi(0)\rangle = \langle\Psi(t)|\hat{Q}_s|\Psi(t)\rangle \\ &= \langle\hat{Q}_s\rangle(t)\end{aligned}$$

For the Schrödinger Picture, we have an eqn. governing the time evolution of the state:

$$i\hbar\frac{d}{dt}|\Psi(t)\rangle = \hat{H}|\Psi\rangle$$

What is the analogous eqn for the Heisenberg Picture?

we need eqn.s governing operator evolution.

For $\hat{Q}_H(t)$, we have: $\hat{Q}_H(t) = \hat{U}^+(a)\,\hat{Q}_H(t)\,\hat{U}(a) = \hat{Q}_H(t+a)$

The infinitesimal form is:

$$\begin{aligned}\hat{U}^+(\epsilon)\,\hat{Q}_H(t)\,\hat{U}(\epsilon) &= \hat{Q}_H(t+\epsilon) \approx \hat{Q}_H(t) + \epsilon\frac{d\hat{Q}_H(t)}{dt} \\ \left(1+\frac{i\epsilon}{\hbar}\hat{H}\right)\hat{Q}_H(t)\left(1-\frac{i\epsilon}{\hbar}\hat{H}\right) &\approx \hat{Q}_H(t) + \epsilon\frac{d\hat{Q}_H(t)}{dt} \\ \hat{Q}_H(t) + \tfrac{i}{\hbar}\epsilon[\hat{H},\hat{Q}_H(t)] &\approx \hat{Q}_H(t) + \epsilon\frac{d\hat{Q}_H(t)}{dt}\end{aligned}$$

& we have

$$\frac{d\hat{Q}_H(t)}{dt} = \frac{i}{\hbar}[\hat{H},\hat{Q}_H(t)]$$

the "Heisenberg eqns of motion."

Example: Harmonic Oscillator

$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$ w/ e-states $|n\rangle$: $\hat{H}|n\rangle = \underbrace{(n+\tfrac{1}{2})\hbar\omega}_{\equiv E_n}|n\rangle$

there are raising + lowering op.s here: $a_\pm \sim \hat{x} \pm i\hat{p}$ w/

$$\hat{a}_+|n\rangle = \sqrt{n+1}\,|n+1\rangle\,,\quad \hat{a}_-|n\rangle = \sqrt{n}\,|n-1\rangle$$

now:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(a_+ + a_-)\,,$$

we'll compute $\langle\Psi(t)|\hat{x}|\Psi(t)\rangle$ in the Schrödinger Picture for

$|\Psi(0)\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$, then

$$|\Psi(t)\rangle = e^{-iE_0t/\hbar}\cos\theta|0\rangle + e^{-iE_1t/\hbar}\sin\theta|1\rangle$$

and

$$\hat{x}|\psi(t)\rangle = \left(\frac{\hbar}{2m\omega}\right)^{1/2}\left(e^{-iE_0t/\hbar}\cos\Theta|1\rangle + e^{-iE_1t/\hbar}\sin\Theta\left(|0\rangle + \sqrt{2}|2\rangle\right)\right)$$

w/

$$\langle\psi(t)|\hat{x}|\psi(t)\rangle = \left(\frac{\hbar}{2m\omega}\right)^{1/2}\cos\Theta\sin\Theta\underbrace{\left[e^{-i(E_0-E_1)t/\hbar} + e^{i(E_0-E_1)t/\hbar}\right]}_{=2\cos((E_0-E_1)t/\hbar)}$$

$$= \left(\frac{2\hbar}{m\omega}\right)^{1/2}\cos\Theta\sin\Theta\cdot\cos(\omega t)$$

($E_0 - E_1$: $= \hbar\omega$)

For the Heisenberg picture: $|\psi(0)\rangle = \cos\Theta|0\rangle + \sin\Theta|1\rangle$ and

$$\frac{d\hat{x}_H(t)}{dt} = \frac{i}{\hbar}\left[\hat{H}, \hat{x}_H(t)\right]$$

We know that $\hat{H}_H(t) = \hat{U}^\dagger\hat{H}\hat{U} = \hat{H}$

since $[\hat{U},\hat{H}] = 0$ & $\hat{U}^\dagger\hat{U} = 1$, so we have:

$$\frac{d\hat{x}_H(t)}{dt} = \frac{i}{\hbar}\left[\hat{H}_H(t), \hat{x}_H(t)\right]$$

w/ $\hat{H}_H(t) = \frac{1}{2m}\hat{p}_H(t)^2 + \frac{1}{2}m\omega^2\hat{x}_H(t)^2$

& we have: $[x_H(t), \hat{p}_H(t)] = [\hat{U}^\dagger\hat{x}\hat{U}, \hat{U}^\dagger\hat{p}\hat{U}]$

$$= \hat{U}^\dagger\hat{x}\hat{U}\hat{U}^\dagger\hat{p}\hat{U} - \hat{U}^\dagger\hat{p}\hat{U}\hat{U}^\dagger\hat{x}\hat{U}$$

$$= \hat{U}^\dagger[\hat{x},\hat{p}]\hat{U} = i\hbar \checkmark$$

$$[\hat{p}_H(t)^2, \hat{x}_H(t)] = -2i\hbar\,\hat{p}_H(t)$$

giving $[\hat{H}_H(t), \hat{x}_H(t)] = -\frac{i\hbar}{m}\hat{p}_H(t)$,

$$\frac{d\hat{x}_H(t)}{dt} = \frac{1}{m}\hat{p}_H(t) \qquad \left(CM: \ \frac{dx}{dt} = \frac{\partial H}{\partial p} = p/m\right)$$

we also have:

$$\frac{d\hat{p}_H(t)}{dt} = \frac{i}{\hbar}\left[\hat{H}_H(t), \hat{p}_H(t)\right] = \frac{i}{\hbar}\left[\tfrac{1}{2}m\omega^2\hat{x}_H(t)^2, \hat{p}_H(t)\right]$$

$$= \tfrac{1}{2}m\omega^2\, i/\hbar\,\underbrace{\left[\hat{x}_H(t)^2, \hat{p}_H(t)\right]}_{=2i\hbar\,\hat{x}_H(t)}$$

so $\frac{d\hat{p}_H(t)}{dt} = -m\omega^2\hat{x}_H(t)$ $\left(CM: \ \frac{dp}{dt} = -\frac{\partial H}{\partial x} = m\omega^2 x\right)$

we can solve $\frac{d^2\hat{x}_H(t)}{dt^2} = -\omega^2\hat{x}_H(t)$

to get: $\hat{x}_H(t) = \hat{A}\cos(\omega t) + \hat{B}\sin(\omega t)$

$$\hat{p}_H(t) = -m\omega\left(\hat{A}\sin(\omega t) - \hat{B}\cos(\omega t)\right)$$

How should we pick $\hat{A}$ & $\hat{B}$? Initial conditions

$\hat{x}_H(0) = \hat{U}^\dagger(0)\,\hat{x}\,\hat{U}(0) = \hat{x}$ & $\hat{p}_H(0) = \hat{p}$

$\hat{x}_H(0) = \hat{A} = \hat{x}$ $\qquad \hat{p}_H(0) = m\omega\hat{B} = \hat{p}$

$$\hat{x}_H(t) = \hat{x}\cos(\omega t) + \frac{\hat{p}}{m\omega}\sin(\omega t).$$

To compute $\hat{x}_H(t)\underbrace{|\Psi(0)\rangle}_{=\cos\Theta|0\rangle+\sin\Theta|1\rangle}$

we need $\hat{p}|\Psi(0)\rangle$ - note that

$$\hat{p} = i\left(\frac{\hbar m\omega}{2}\right)^{1/2}(\hat{a}_+ - \hat{a}_-)$$

so

$$\hat{p}|\Psi(0)\rangle = i\left(\frac{\hbar m\omega}{2}\right)^{1/2}\left(\cos\Theta|1\rangle + \sin\Theta(\sqrt{2}|2\rangle - |0\rangle)\right)$$

+

$$\langle\Psi(0)|\hat{p}|\Psi(0)\rangle = i\left(\frac{\hbar m\omega}{2}\right)^{1/2}\underbrace{(\sin\Theta\cos\Theta - \sin\Theta\cos\Theta)}_{=0}$$

$$\hat{x}|\Psi(0)\rangle = \left(\frac{\hbar}{2m\omega}\right)^{1/2}\left(\cos\Theta|1\rangle + \sin\Theta(\sqrt{2}|2\rangle + |0\rangle)\right)$$

w/

$$\langle\Psi(0)|\hat{x}|\Psi(0)\rangle = 2\left(\frac{\hbar}{2m\omega}\right)^{1/2}\sin\Theta\cos\Theta$$

Finally,

$$\langle\Psi(0)|\hat{x}_H(t)|\Psi(0)\rangle = \sqrt{\frac{2\hbar}{m\omega}}\cos\Theta\sin\Theta\cos(\omega t)\ \checkmark$$

Time-Independent Perturbation Theory

Problem: Given $U(x)$, find $\{\Psi_j(x)\}_{j=0}^{\infty}$, $\{E_j\}_{j=0}^{\infty}$ such that

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi_j(x)}{dx^2} + U(x)\Psi_j(x) = E_j\Psi_j(x)$$

In general, this is hard to do.

We can get partial answers if $U(x) = U^0(x) + \epsilon U'(x)$ for $\epsilon \ll 1$, + we know the states satisfying

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi_j^0(x)}{dx^2} + U^0(x)\Psi_j^0(x) = E_j^0\Psi_j^0(x)$$

The linear algebra problem we are solving is:

If you know $\{\vec{v}_j^{\,0}\}_{j=1}^{n}$ + $\{\lambda_j^0\}_{j=1}^{n}$ w/

$$\mathbb{A}^0\vec{v}_j^{\,0} = \lambda_j^0\vec{v}_j^{\,0}$$

Find an approximation to the $\{\vec{v}_j\}_{j=1}^{n}$, $\{\lambda_j\}_{j=1}^{n}$ satisfying

$$(\mathbb{A}^0 + \epsilon\mathbb{A}')\vec{v}_j = \lambda_j\vec{v}_j.$$