Last Time

we were generating the rotation selection rules for vectors:

using $[\hat{L}_z, \hat{V}_z] = 0$,

$[\hat{L}_z, \hat{V}_\pm] = \pm\hbar \hat{V}_\pm$,

in $\langle l'm'|[\;,\;]|lm\rangle$

we learned that

$\hbar(m'-m)\langle l'm'|\hat{V}_z|lm\rangle = 0$

so $\langle l'm'|\hat{V}_z|lm\rangle = 0$

unless $m'=m$.

and

$\hbar(m'-m\mp1)\langle l'm'|\hat{V}_\pm|lm\rangle = 0$

giving the pair:

$\hbar(m'-(m+1))\langle l'm'|\hat{V}_+|lm\rangle = 0$

& $\langle l'm'|\hat{V}_+|lm\rangle = 0$

unless $m'=m+1$.

$\hbar(m'-(m-1))\langle l'm'|\hat{V}_-|lm\rangle = 0$

so $\langle l'm'|\hat{V}_-|lm\rangle = 0$

unless $m'=m-1$.

Vector Selection Rules

From "the rest" of the commutators ($[\hat{L}_\pm, \hat{V}_\pm]$, $[\hat{L}_\pm, \hat{V}_\mp]$, $[\hat{L}_\pm, \hat{V}_z]$) we do the same thing, then all together, we end up w/:

$$\langle l'm'|\hat{V}_+|lm\rangle = -\sqrt{2}\, C^{l1l'}_{m1m'} \langle l'\|V\|l\rangle$$

$$\langle l'm'|\hat{V}_z|lm\rangle = C^{l1l'}_{m0m'} \langle l'\|V\|l\rangle$$

$$\langle l'm'|\hat{V}_-|lm\rangle = \sqrt{2}\, C^{l1l'}_{m-1m'} \langle l'\|V\|l\rangle$$

where $\langle l'\|V\|l\rangle$ is the "reduced matrix elt." independent of m & m'.

- $C^{l_1 l_2 L}_{m_1 m_2 M} = 0$ if $m_1+m_2 \neq M$

  ($|l_1 l_2 m_1 m_2\rangle$ are e-states of $\hat{L}_z = \hat{L}_{1z} + \hat{L}_{2z}$, so $M = m_1+m_2$ is always the case)

- the total L go from l_1+l_2 down to $|l_1-l_2|$.

  ex. if $l_1 = 3$, $l_2 = 1$, we can get total $L = 4, 3, 2$.

  $C^{l_1 l_2 L}_{m_1 m_2 M} = 0$ if $L \neq l_1+l_2 \to |l_1-l_2|$

In terms of the coefficients appearing above,

$$C^{l1l'}_{m-m'} = 0 \quad \text{unless } l' = l+1,\ l,\ l-1$$

so we only get non-zero results if $\Delta m = 0, \pm1$, $\Delta l = 0, \pm1$.

How do we use these relations? Aside from avoiding calculation when $C^{l1l'}_{m-m'} = 0$ (& you should always do that), we just need to compute

$$\langle l'\|V\|l\rangle$$

(or, for more general situations like hydrogen, $\langle n'l'\|V\|nl\rangle$)

Suppose you were interested in states like

$$\underset{n\ l\ m}{|322\rangle},\ |321\rangle,\ |320\rangle,\ |32-1\rangle,\ |32-2\rangle$$

& their linear combinations — you have some concrete vector operator in mind: $\vec{r}_e$, $\vec{E}$, $\vec{B}$, etc. call it $\vec{V}$.

to get $\langle 32\|V\|32\rangle$ pick "simple"* states, like $m=0,1$, & compute

$$\langle 321|\hat{V}_z|321\rangle$$

From the selection rules,

$$\langle 321|\hat{V}_z|321\rangle = C^{212}_{101}\langle 32\|V\|32\rangle, \quad C = \sqrt{1/6}$$

then

$$\langle 32\|V\|32\rangle = \sqrt{6}\,\langle 321|\hat{V}_z|321\rangle$$

* why not use $|320\rangle$ by itself? $\langle 320|\hat{V}_z|320\rangle = ?$ what type of vector has non-zero $\langle 321|\hat{V}_z|321\rangle$?

Time Translation

In CM, we had $\bar{x} = x + \epsilon \frac{\partial J}{\partial p}$, $\bar{p} = p - \epsilon \frac{\partial J}{\partial x}$

w/ $\frac{dJ}{dt} = \{J, H\}_{P.B.}$

If $\{J, H\}_{PB} = 0$, $\frac{dJ}{dt} = 0$ & J is a constant (Noether's thm).

The choice $J = H$ has $\{H, H\}_{P.B.} = 0$ automatically, "so" $\frac{dH}{dt} = 0$

The transformation generated by H is:

$$\bar{x}(t) = x(t) + \epsilon \frac{\partial H}{\partial p} = x(t) + \epsilon \dot{x}(t) = x(t+\epsilon)$$

$$\bar{p}(t) = p(t) - \epsilon \frac{\partial H}{\partial x} = p(t) + \epsilon \dot{p}(t) = p(t+\epsilon)$$

so "H generates time translation in CM"

How about quantum mechanics?

$$\psi(x, t+\epsilon) = \psi(x,t) + \epsilon \frac{\partial \psi}{\partial t}, \quad \&$$

Schrödinger's eqn. is:

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi \Rightarrow \epsilon \frac{\partial \psi}{\partial t} = \frac{\epsilon}{i\hbar} \hat{H}\psi,$$

$$\& \quad \psi(x, t+\epsilon) = \psi(x,t) + \epsilon \frac{\partial \psi}{\partial t}$$

$$= \underbrace{\left(1 - \frac{i}{\hbar}\epsilon \hat{H}\right)}_{\text{first term of } e^{-\frac{i}{\hbar}\epsilon \hat{H}}} \psi(x,t)$$

then the time translation operator is:

$$\hat{U}(t) = e^{-\frac{i}{\hbar}t\hat{H}}$$

and $\psi(x,t) = \hat{U}(t)\psi(x,0)$ (or $|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle$)

Notice that when $\hat{U}$ acts on an eigenstate of $\hat{H}$: $\hat{H}|\psi\rangle = E|\psi\rangle$,

$$\hat{U}|\psi\rangle = e^{-\frac{i}{\hbar}t\hat{H}}|\psi\rangle = \sum_{j=0}^{\infty} \frac{1}{j!}\left(-\frac{i}{\hbar}t\hat{H}\right)^j |\psi\rangle$$

$$\left(\hat{H}^j|\psi\rangle = E^j|\psi\rangle\right)$$

$$= \sum_{j=0}^{\infty} \frac{1}{j!}\left(-\frac{i}{\hbar}tE\right)^j |\psi\rangle$$

$$= e^{-iEt/\hbar}|\psi\rangle$$

so that if $\psi(x)$ solves $\hat{H}\psi = E\psi$, then

$\psi(x,t) = e^{-iEt/\hbar}\psi(x)$ is the solution to Schrödinger's eqn., working in position basis

For $\hat{H}\psi_j(x) = E_j\psi_j(x)$, &

$$\psi(x,0) = \sum_{j=0}^{\infty} c_j \psi_j(x)$$

the initial wavefunction decomposed into eigenstates of $\hat{H}$,

$$e^{-\frac{i}{\hbar}t\hat{H}}\psi(x,0) = \sum_{j=0}^{\infty} c_j e^{-iE_j t/\hbar}\psi_j(x)$$

$$= \psi(x,t)$$

a familiar confirmation of the time-translation operator's action.

From $\frac{d\langle\hat{H}\rangle}{dt} = \frac{i}{\hbar}\langle[\hat{H},\hat{H}]\rangle = 0$
we see that the expectation value of $\hat{H}$ (the ___?) is a constant.

What happens to operators $\hat{Q}(\hat{x},\hat{p})$ under time translation?

Unclear what to make of the transformed operator... (no explicit t-dep. in $\hat{x}, \hat{p}$)

What is $\hat{\hat{x}} = \hat{U}^+ \hat{x}\, \hat{U}$?

Try it out w/ $|\Psi(t)\rangle$:

$$\langle\Psi(t)|\hat{x}|\Psi(t)\rangle = \langle\Psi(0)|\underbrace{\hat{U}^+\hat{x}\,\hat{U}}_{=\hat{\hat{x}}}|\Psi(0)\rangle$$

$$= \langle\Psi(0)|\,\hat{\hat{x}}\,|\Psi(0)\rangle$$

so that $\hat{\hat{x}}$ acting on the state $|\Psi(0)\rangle$ is the same as $\hat{x}$ acting on the state $|\Psi(t)\rangle$

The "time-translation" is now built into the operator $\hat{\hat{x}}$.

Similarly, for a general $\hat{Q}(\hat{x},\hat{p})$, we have:

$$\hat{\hat{Q}} = \hat{U}^+\hat{Q}\,\hat{U} \quad \&$$

$$\langle\Psi(0)|\,\hat{\hat{Q}}\,|\Psi(0)\rangle = \langle\Psi(t)|\,\hat{Q}\,|\Psi(t)\rangle$$

In the "Schrödinger picture" the operators are t-indep., & the wave-function moves in time.

In the "Heisenberg picture," the wave function stays the same, $|\Psi(0)\rangle$, & the operators evolve in time. Such operators are called "Heisenberg operators"

$$\underbrace{\hat{Q}_H(t)}_{\text{Heisenberg}} = \hat{U}^+(t)\underbrace{\hat{Q}_S}_{\text{Schrödinger}}\hat{U}(t)$$

What happens to Schrödinger's eqn., governing the temporal evolution of the wave function? → It gets replaced w/ eqn.'s governing the time evolution of the operators - we can get these from:

$$\hat{U}^+(\epsilon)\hat{Q}_H(t)\hat{U}(\epsilon) = \hat{Q}_H(t+\epsilon) \approx \hat{Q}_H(t) + \epsilon\frac{d\hat{Q}_H}{dt}$$

the infinitesimal form of $\hat{U}(\epsilon) \approx 1 - \frac{i}{\hbar}\epsilon\hat{H}$

so

$$\hat{Q}_H(t) + \epsilon\frac{d\hat{Q}_H}{dt} \approx \left(1+\tfrac{i}{\hbar}\epsilon\hat{H}\right)\hat{Q}_H(t)\left(1-\tfrac{i}{\hbar}\epsilon\hat{H}\right)$$

$$= \hat{Q}_H(t) + \epsilon\,\tfrac{i}{\hbar}[\hat{H},\hat{Q}_H]$$

giving $\frac{dQ_H(t)}{dt} = \frac{i}{\hbar}[\hat{H},\hat{Q}_H(t)]$