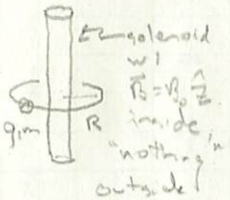
Last Time



$$E=\frac{\hbar^2}{2mR^2}\left(\pm n-\frac{qAR}{\hbar}\right)^2$$

w/ $A=A(R)$, the value of the vector potential at the particle location

$\vec{B}=\nabla\times\vec{A}$ — integrate over disk of radius R:

$$\int_D \vec{B}\cdot d\vec{a}=\int_D(\nabla\times\vec{A})\cdot d\vec{a}$$

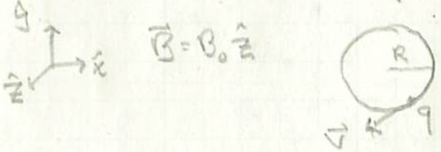
$$\underbrace{\Phi}_{\text{magnetic flux through the disk}}=\oint_{\partial D}\vec{A}\cdot d\vec{l}$$

$$=A(R)\cdot 2\pi R$$

so

$$E=\frac{\hbar^2}{2mR^2}\left(\pm n-\underbrace{\frac{q\Phi}{2\pi\hbar}}_{\text{insensitive to gauge choice}}\right)^2$$

Motion in a Uniform Magnetic Field



the radius of the particle motion is:

$$\frac{mv^2}{R}=qvB_0\Rightarrow R=\frac{mv}{qB_0}$$

the period of the motion is:

$$vT=2\pi R\Rightarrow T=\frac{2\pi m}{qB_0}$$

so the "cyclotron frequency" is

$$\omega=\frac{2\pi}{T}=\frac{qB_0}{m}$$

for the quantum mechanical version, we need $\vec{A}$:

$$(\nabla\times\vec{A})_z=\frac{\partial A^y}{\partial x}-\frac{\partial A^x}{\partial y}\qquad \text{take } \vec{A}=B_0x\hat{y}$$

then the piece of the Schrödinger eqn. that looks like

$$\frac{1}{2m}q^2A^2\psi=\frac{1}{2m}\underbrace{\frac{q^2B_0^2}{m^2}}_{=\omega^2}\cdot m^2x^2$$

$$=\tfrac{1}{2}m\omega^2x^2$$

like a harmonic oscillator "potential energy"

quantized energies end up being the same as in that case:

$$E=(n+\tfrac{1}{2})\hbar\omega$$

Ehrenfest's Theorem

We can say more about motion in a magnetic field using Ehrenfest's theorem — for an operator Q, we have:

$$\frac{d}{dt}\langle Q\rangle=\frac{i}{\hbar}\langle[H,Q]\rangle$$

for magnetism, $H=\frac{1}{2m}(\vec{p}-q\vec{A})\cdot(\vec{p}-q\vec{A})$

& define the operator $\vec{v}=\frac{1}{m}(\vec{p}-q\vec{A})$ (from last time, we saw $\vec{p}=m\vec{v}+q\vec{A}$), then $H=\frac{1}{2}mv^2$, & we have:

$$\frac{d}{dt}\langle\vec{v}\rangle=\frac{i}{\hbar}\langle[H,\vec{v}]\rangle=\frac{i}{\hbar}\cdot\frac{1}{2}m\langle[v^2,\vec{v}]\rangle$$

we'll calculate the commutator $[v^2,v_x]$ as an example.

$$[v^2,v_x]=[v_x^2+v_y^2+v_z^2,v_x]$$

$$=[v_y^2,v_x]+[v_z^2,v_x]$$

In order to compute $[v_y^2, v_x]$, we need to evaluate

$$[v_x, v_y] = \frac{1}{m^2}[p_x - qA_x,\ p_y - qA_y] \qquad (*)$$

$$= \frac{1}{m^2}\left\{\cancel{[p_x, p_y]} - q[p_x, A_y] - q[A_x, p_y] + q^2\cancel{[A_x, A_y]}\right\}$$

and for this, we need $[p_x, A_y]$ ... use a test function

$$[p_x, A_y] f(x,y,z) = \frac{\hbar}{i}\frac{\partial}{\partial x}(A_y f) - A_y\frac{\hbar}{i}\frac{\partial f}{\partial x}$$

$$= \frac{\hbar}{i}\frac{\partial A_y}{\partial x} f \Rightarrow [p_x, A_y] = \frac{\hbar}{i}\frac{\partial A_y}{\partial x}$$

& sim. $[A_x, p_y] = -[p_y, A_x] = -\frac{\hbar}{i}\frac{\partial A_x}{\partial y}$.

Using these in (*) gives:

$$[v_x, v_y] = \frac{-q\hbar}{im^2}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) = \frac{-q\hbar}{im^2}B_z$$

(Aside: $[A,B] = C \Rightarrow [A^2, B] = AC + CA$)

So

$$[v_y^2, v_x] = \frac{q\hbar}{im^2}(v_y B_z + B_z v_y)$$

& similarly

$$[v_z^2, v_x] = \frac{-q\hbar}{im^2}(v_z B_y + B_y v_z)$$

then $[v^2, v_x] = \frac{q\hbar}{im^2}(v_y B_z + B_z v_y - v_z B_y - B_y v_z)$

$$= \frac{q\hbar}{im^2}\left(+(\vec{v}\times\vec{B})_x - (\vec{B}\times\vec{v})_x\right)$$

we get $[v^2, \vec{v}] = \frac{q\hbar}{im^2}(\vec{v}\times\vec{B} - \vec{B}\times\vec{v})$

&, finally:

$$\frac{d}{dt}\langle\vec{v}\rangle = \frac{i}{\hbar}\cdot\frac{1}{2}m\cdot\frac{q\hbar}{im^2}\langle\vec{v}\times\vec{B} - \vec{B}\times\vec{v}\rangle$$

$$= \frac{1}{2m}\langle q\vec{v}\times\vec{B} - q\vec{B}\times\vec{v}\rangle \qquad (+)$$

it will not, in general, be the case that $\vec{B}\times\vec{v} = -\vec{v}\times\vec{B}$, so we are done.

For a uniform field, though: $\langle\vec{v}\times\vec{B}\rangle = \langle\vec{v}\rangle\times\vec{B}$

$$\langle\vec{B}\times\vec{v}\rangle = \vec{B}\times\langle\vec{v}\rangle = -\langle\vec{v}\rangle\times\vec{B}$$

& $m\frac{d}{dt}\langle v\rangle = q\langle\vec{v}\rangle\times\vec{B}$

the expectation value of velocity behave just "like" the classical velocity.

Last Classical Calculation

let's do the classical version of this calculation

$$\frac{d\vec{v}}{dt} = \{\vec{v}, H\}_{P.B.}$$

the Poisson bracket for functions of $x, y, z, p_x, p_y + p_z$ is

$$\{f, g\}_{P.B.} = \sum_{i=1}^{3} \left(\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x^i} \right)$$

we have $\vec{v} = \frac{1}{m}(\vec{p} - q\vec{A})$, + we'll focus on V_x again to see the pattern. $H = \frac{1}{2m}(\vec{p}-q\vec{A})\cdot(\vec{p}-q\vec{A})$ calculations, so

$$= \frac{1}{2m}(p^2 - 2q\vec{A}\cdot\vec{p} + q^2A^2)$$

$$\{v_x, H\} = \frac{1}{2m^2}\left(-q\frac{\partial A_x}{\partial x}(2p_x - 2qA_x) - q\frac{\partial A_x}{\partial y}(2p_y - 2qA_y) - q\frac{\partial A_x}{\partial z}(2p_z - 2qA_z)\right.$$

$$\left. - 1\cdot\left(-2q\frac{\partial}{\partial x}(\vec{A}\cdot\vec{p}) + q^2\frac{\partial}{\partial x}A^2\right)\right)$$

$$= \frac{1}{2m^2}\left(-2q\nabla A_x\cdot(\vec{p}-q\vec{A}) + 2q\frac{\partial\vec{A}}{\partial x}\cdot(\vec{p}-q\vec{A})\right)$$

$$= -q/m\left(\underbrace{\left(\nabla A_x - \frac{\partial\vec{A}}{\partial x}\right)\cdot\vec{v}}_{= -(\vec{v}\times(\nabla\times\vec{A}))_x}\right) = q/m\,(\vec{v}\times\vec{B})_x$$

So we recover $\frac{d\vec{v}}{dt} = \frac{q}{m}\vec{v}\times\vec{B}$

? How might we get the full QM version (+) using this approach ?