Addition of Angular Momenta

Suppose we have: $|\ell m\rangle$ w/

$$\hat{L}^2|\ell m\rangle = \hbar^2 \ell(\ell+1)|\ell m\rangle$$

$$\hat{L}_z|\ell m\rangle = \hbar m|\ell m\rangle$$

If we have 2 angular momenta (could be 2 particles, or 2 different sources of momentum)

$$\hat{L}_1^2|\ell_1 \ell_2 m_1 m_2\rangle = \hbar^2 \ell_1(\ell_1+1)|\ell_1 \ell_2 m_1 m_2\rangle$$

$$\hat{L}_2^2|\ell_1 \ell_2 m_1 m_2\rangle = \hbar^2 \ell_2(\ell_2+1)|\ell_1 \ell_2 m_1 m_2\rangle$$

$$\hat{L}_{1z}|\ell_1 \ell_2 m_1 m_2\rangle = \hbar m_1|\ell_1 \ell_2 m_1 m_2\rangle$$

$$\hat{L}_{2z}|\ell_1 \ell_2 m_1 m_2\rangle = \hbar m_2|\ell_1 \ell_2 m_1 m_2\rangle$$

we can form the operator: $\hat{\vec{L}} = \hat{\vec{L}}_1 + \hat{\vec{L}}_2$, then

$$\hat{L}_z|\ell_1 \ell_2 m_1 m_2\rangle = (\hat{L}_{1z} + \hat{L}_{2z})|\ell_1 \ell_2 m_1 m_2\rangle$$

$$= \hbar(m_1+m_2)|\ell_1 \ell_2 m_1 m_2\rangle$$

so $|\ell_1 \ell_2 m_1 m_2\rangle$ is an eigenstate of $\hat{L}_z$.

But: $\hat{L}^2 = \hat{L}_1^2 + \hat{L}_2^2 + 2\vec{L}_1 \cdot \vec{L}_2$
and

$$\hat{L}^2|\ell_1 \ell_2 m_1 m_2\rangle = [\hbar^2 \ell_1(\ell_1+1) + \hbar^2 \ell_2(\ell_2+1)]|\ell_1 \ell_2 m_1 m_2\rangle$$

$$+ 2\vec{L}_1 \cdot \vec{L}_2|\ell_1 \ell_2 m_1 m_2\rangle$$

? For fixed ℓ_1 & ℓ_2 what eigenstates of $\hat{L}^2$ & $\hat{L}_z$ can we make ?

we'll have $m_1 + m_2$ for $\hat{L}_z$ acting on $|\ell_1 \ell_2 m_1 m_2\rangle$ & $m_1: -\ell_1 \to \ell_1$, $m_2: -\ell_2 \to \ell_2$, so we'll get

$$-\ell_1 - \ell_2 \longrightarrow \ell_1 + \ell_2 \quad \text{values}$$

example: $\ell_1 = 1$ $\ell_2 = 2$, you can make states w/

$$M = -3, -2, -1, 0, 1, 2, 3$$

the associated total angular momentum L could be 3, 2, 1 or 0. (as it turns out, you only get $L = \ell_1 + \ell_2 \to |\ell_1 - \ell_2|$, so here, $L = 3, 2, 1$)

The most general linear combination we can make is:

$$|\psi\rangle = \_\_|\ell_1 \ell_2 -\ell_1 -\ell_2\rangle + \_\_|\ell_1 \ell_2 (-\ell_1+1) -\ell_2\rangle + \_\_|\ell_1 \ell_2 \ell_1 (-\ell_2+1)\rangle$$

$$+ \_\_|\ell_1 \ell_2 (-\ell_1+1)(-\ell_2+1)\rangle + \dots + |\ell_1 \ell_2 \ell_1 \ell_2\rangle$$

and we want $\hat{L}^2|\psi\rangle = \hbar^2 L(L+1)|\psi\rangle$

$$\hat{L}_z|\psi\rangle = \hbar M|\psi\rangle$$

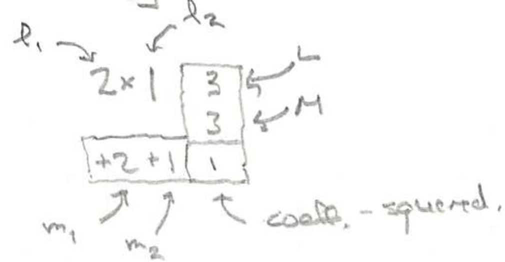
The linear combination that has L & M is:

$$|LM\rangle = \sum_{\substack{m_1 = -\ell_1 \to \ell_1 \\ m_2 = -\ell_2 \to \ell_2}} C^{\ell_1 \ell_2 L}_{m_1 m_2 M}|\ell_1 \ell_2 m_1 m_2\rangle$$

w/ $C^{\ell_1 \ell_2 L}_{m_1 m_2 M} = 0$ if $m_1 + m_2 \neq M$ the Clebsch-Gordon coeffs

Easy to imagine generating such coefficients, but tedious in practice!

Fortunately, tables exist:



For $\ell_1 = 2$, $\ell_2 = 1$, then, we have:

$$|3\,3\rangle = |2\,1\,2\,1\rangle$$

sticking w/ that same $\ell_1 + \ell_2$, the state w/ $L = 2$, $M = 0$ is:

$$|2\,0\rangle = \tfrac{1}{\sqrt{2}}\,|2\,1\,1\,{-1}\rangle - \tfrac{1}{\sqrt{2}}\,|2\,1\,{-1}\,1\rangle$$

etc.

? For $\ell_1 = 2$, $\ell_2 = 2$, what's the $L = 2$, $M = 1$ state:

$$|2\,1\rangle = \;?$$

Selection Rules for Rotations

For a scalar operator $\hat{S}$, a rotation "does nothing":

$$\hat{\hat{S}} = \hat{R}_z^\dagger \hat{S} \hat{R}_z = \hat{S} \Rightarrow [\hat{R}_z, \hat{S}] = 0$$

but then, since $\hat{R}_z \cong 1 - i/\hbar\, \varphi \hat{L}_z$

$$[\hat{R}_z, \hat{S}] = [1 - i/\hbar\,\varphi \hat{L}_z, \hat{S}] = -i/\hbar\,\varphi[\hat{L}_z, \hat{S}] = 0$$

so $[\hat{L}_z, \hat{S}] = 0$, too.

The same is true for $[\hat{L}_x, \hat{S}] = [\hat{L}_y, \hat{S}] = 0$

& then $[\hat{L}^2, \hat{S}] = 0$ & $[\hat{L}_\pm, \hat{S}] = 0$ w/ $\hat{L}_\pm = \hat{L}_x \pm i \hat{L}_y$

Aside for the action of $\hat{L}_+$ on $|\ell m\rangle$, let $|\psi\rangle = \hat{L}_+ |\ell m\rangle$ then

$$\begin{aligned} \hat{L}_z|\psi\rangle &= \hat{L}_z \hat{L}_+ |\ell m\rangle \\ &= (\hat{L}_+ \hat{L}_z + \hbar \hat{L}_+)|\ell m\rangle \\ &= \hat{L}_+(\hbar m + \hbar)|\ell m\rangle \\ &= \hbar(m+1)|\psi\rangle \end{aligned} \qquad \begin{aligned} [\hat{L}_z, \hat{L}_+] &= [\hat{L}_z, \hat{L}_x] + i[\hat{L}_z, \hat{L}_y] \\ &= i\hbar(\hat{L}_y - i\hat{L}_x) \\ &= \hbar \hat{L}_+ \end{aligned}$$

so $\hat{L}_+|\ell m\rangle$ is an eigenstate of $\hat{L}_z$ w/ e-val $\hbar(m+1)$

If $m+1 > \ell$, $|\psi\rangle = 0$, you can't go past the top rung. $\hat{L}_+|\ell m\rangle \sim |\ell\, m+1\rangle$ w/ norm. constant:

$$\hat{L}_+|\ell m\rangle = A_\ell^m |\ell\, m+1\rangle \quad A_\ell^m = \hbar\left(\ell(\ell+1) - m(m+1)\right)^{1/2}$$

sim.

$$\hat{L}_-|\ell m\rangle = B_\ell^m |\ell\, m-1\rangle \quad B_\ell^m = \hbar\left(\ell(\ell+1) - m(m-1)\right)^{1/2}$$

Back to our regularly scheduled program:

Take each of the: $[\hat{L}_z,\hat{S}]=0$, $[\hat{L}^2,\hat{S}]=0$, $[\hat{L}_\pm,\hat{S}]=0$ & sandwich between $\langle l'm'|$ & $|lm\rangle$:

$$\langle l'm'|[\hat{L}_z,\hat{S}]|lm\rangle = \underbrace{\langle l'm'|\hat{L}_z}_{=\hbar m'\langle l'm'|}\hat{S}|lm\rangle - \langle l'm'|\hat{S}\underbrace{\hat{L}_z|lm\rangle}_{=\hbar m|lm\rangle}$$

$$0 = \hbar m'\langle l'm'|\hat{S}|lm\rangle - \hbar m\langle l'm'|\hat{S}|lm\rangle$$

so $\hbar(m'-m)\langle l'm'|\hat{S}|lm\rangle=0$ either $m'=m$, or $\langle l'm'|\hat{S}|lm\rangle=0$

Similarly, for $[\hat{L}^2,\hat{S}]=0$,

$$\langle l'm'|[\hat{L}^2,\hat{S}]|lm\rangle = \langle l'm'|\hat{L}^2\hat{S}|lm\rangle - \langle l'm'|\hat{S}\hat{L}^2|lm\rangle$$

$$0 = \hbar^2\big(l'(l'+1)-l(l+1)\big)\langle l'm'|\hat{S}|lm\rangle$$

and if $l'\neq l$, we'll have $\langle l'm'|\hat{S}|lm\rangle=0$

For $[\hat{L}_+,\hat{S}]=0$, we need to evaluate:

$$\hat{L}_+|lm\rangle = A_l^m|l\,m+1\rangle$$

& $\langle l'm'|\hat{L}_+ = \langle \hat{L}_+^\dagger l'm'|$

w/ $\hat{L}_+^\dagger = \hat{L}_x - i\hat{L}_y = \hat{L}_-$

$= B_{l'}^{m'}\langle l'\,m'-1|$

Then $\langle l'm'|[\hat{L}_+,\hat{S}]|lm\rangle=0$ gives

$$B_{l'}^{m'}\underbrace{\langle l'\,m'-1|\hat{S}|l\,m\rangle}_{\substack{=0\text{ unless}\\ m'-1=m\\ \&\ l'=l}} - A_l^m\underbrace{\langle l'm'|\hat{S}|l\,m+1\rangle}_{\substack{=0\text{ unless}\\ m'=m+1\\ \&\ l'=l}}=0$$

$$B_l^{m+1}\langle lm|\hat{S}|lm\rangle = A_l^m\langle l\,m+1|\hat{S}|l\,m+1\rangle$$

& $B_l^{m+1} = \hbar\big(l(l+1)-(m+1)m\big)^{1/2} = A_l^m$

so

$$\langle lm|\hat{S}|lm\rangle = \langle l\,m+1|\hat{S}|l\,m+1\rangle$$

$\langle lm|\hat{S}|lm\rangle$ is independent of m

Shorthand for these observations:

$$\langle l'm'|\hat{S}|lm\rangle = \delta_{ll'}\delta_{mm'}\underbrace{\langle l\|\hat{S}\|l\rangle}_{\#\text{ that depends on }l}$$