Lost Time

We defined a vector to be any set of 3 quantities that responded to rotations "like the coords do":

ex. $$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$ (for z-axis rot.)

induces

$$\begin{pmatrix} \bar{E}^x \\ \bar{E}^y \\ \bar{E}^z \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E^x \\ E^y \\ E^z \end{pmatrix}$$

so $\vec{E}$ is a vector (under rotations)

For the infinitesimal ($\theta \ll 1$) form, we have

$$\bar{x} = x - \overbrace{\theta y}^{=\frac{\partial J}{\partial p_x}} \qquad \bar{y} = y + \overbrace{\theta x}^{=\frac{\partial J}{\partial p_y}} \qquad \bar{z} = z \qquad (*)$$

and for vectors like $\vec{E}$:

$$\bar{E}^x = E^x - \theta E^y \qquad \bar{E}^y = E^y + \theta E^x \qquad \bar{E}^z = E^z \qquad (\circ)$$

The inf. coord. transformation in $(*)$ is generated by
$J = \theta(xp_y - yp_x) = \theta(\vec{r}\times\vec{p})_z = \theta L_z$

We then showed that: $\{L_i, L_j\}_{P.B.} = \epsilon_{ijk} L_k$
which implies

$$[\hat{L}_i, \hat{L}_j] = i\hbar\,\epsilon_{ijk}\hat{L}_k$$

But $\{L_i, L_j\}_{P.B.} = \epsilon_{ijk}L_k$ leads us to expect that

$$\{L_i, V_j\}_{P.B.} = \epsilon_{ijk} V_k \quad \text{for a vector } \vec{V}.$$

example: $$\{\underbrace{\theta L_z}_{xp_y - yp_x}, \underbrace{E^x}_{\text{func. of } \vec{r}, \text{ not } \vec{p}.}\} = \theta\left(-\frac{\partial L_z}{\partial p_x}\frac{\partial E^x}{\partial x} - \frac{\partial L_z}{\partial p_y}\frac{\partial E^x}{\partial y} - \frac{\partial L_z}{\partial p_z}\frac{\partial E^x}{\partial z}\right)$$

$$= -\theta\left(-y\frac{\partial E^x}{\partial x} + x\frac{\partial E^x}{\partial y}\right)$$

$$\approx -E^x(x-\theta y, y+\theta x, z) + E^x(x,y,z)$$

$$\approx -\bar{E}^x + E^x \approx \theta E^y \quad \text{from } (\circ)$$

so $\{L_z, E^x\} = E^y \Rightarrow \{L_i, E_j\} = \epsilon_{ijk}E_k$

Check: $3 \quad 1 \qquad \epsilon_{312}E_2 = E_2$ ✓

Just as: $\{L_i, L_j\}_{P.B.} = \epsilon_{ijk}L_k \xrightarrow{QM} [\hat{L}_i, \hat{L}_j] = i\hbar\,\epsilon_{ijk}\hat{L}_k$

$\{L_i, E_j\}_{P.B.} = \epsilon_{ijk}E_k \xrightarrow{QM} [\hat{L}_i, \hat{E}_j] = i\hbar\,\epsilon_{ijk}\hat{E}_k$

or, for any vector $\vec{V}$ under rotations:

$$\{L_i, V_j\}_{P.B.} = \epsilon_{ijk}E_k \xrightarrow{QM} [\hat{L}_i, \hat{V}_j] = i\hbar\,\epsilon_{ijk}\hat{V}_k$$

Rotation Operator in QM

For translations: $\hat{T}(a)\psi(x) = \psi(x-a)$, we had

$$\hat{T}(a) = e^{-i/\hbar\, a\hat{p}} \approx (1 - i/\hbar\, a\hat{p}) \quad \text{for small } a.$$

similarly, we expect $\hat{R}_z(\theta) = e^{-i/\hbar\,\theta\hat{L}_z} \approx (1 - i/\hbar\,\underset{\theta \ll 1}{\theta}\hat{L}_z)$

check: $\hat{R}_z(\epsilon)\psi(x,y,z) \approx \psi - i/\hbar\left(x\frac{\hbar}{i}\frac{\partial}{\partial y} - y\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\epsilon\psi$

$$= \psi - \epsilon x\frac{\partial\psi}{\partial y} + \epsilon y\frac{\partial\psi}{\partial x}$$

or $\hat{R}_z(\epsilon)\psi(x,y,z) \approx \psi(x+\epsilon y, y-\epsilon x, z)$

$$\approx \psi(x\cos(-\epsilon) - \sin(-\epsilon)y,\ y\cos(\epsilon) + x\cos(-\epsilon), z)$$

a rotation through $\theta = -\epsilon$

How about the operator transformations – what happens to $\hat{Q}(\hat{r},\hat{p})$ under a rotation? (sticking w/ z-axis rot. as an example)

$$\hat{\bar{Q}} = \hat{R}_z^\dagger\hat{Q}\hat{R}_z \approx (1 + i/\hbar\,\theta\hat{L}_z)\hat{Q}(1 - i/\hbar\,\theta\hat{L}_z)$$

$$\approx \hat{Q} + i/\hbar\,\theta[\hat{L}_z,\hat{Q}] + O(\theta^2)$$

example: $\hat{Q} = \hat{x}$, $\hat{\bar{x}} = \hat{x} + i/\hbar\,\theta[\underbrace{\hat{L}_z}_{=\hat{x}\hat{p}_y - \hat{y}\hat{p}_x},\hat{x}]$

$$= \hat{x} - i/\hbar\,\theta\left(\cancel{[\hat{x}\hat{p}_y,\hat{x}]} - [\hat{y}\hat{p}_x,\hat{x}]\right)$$

$$= \hat{x} - i/\hbar\,\theta\left(\hat{y}\underbrace{[\hat{p}_x,\hat{x}]}_{=-i\hbar}\right)$$

$$= \hat{x} - \theta\hat{y}$$

and sim. $\hat{\bar{y}} = \hat{y} + \theta\hat{x}$, $\hat{\bar{z}} = \hat{z}$

so the QM vector operators respond to a rotation (about the z-axis) in the same way as the CM vector components.

The association:

$$\{L_i, V_j\}_{PB} = \epsilon_{ijk}V_k \xrightarrow{QM} [\hat{L}_i,\hat{V}_j] = i\hbar\,\epsilon_{ijk}\hat{V}_k \quad (\square)$$

is, then, justified.

Spherically Symmetric Hamiltonians (QM)

For $\hat{H} = \hat{p}^2/2m + U(r)$

we showed that for vector operator $\vec{\hat{p}}$,

$[\hat{L}_z,\hat{p}^2] = 0$ using $(\square)$ (w/ $\vec{\hat{V}} = \vec{\hat{p}}$)

to see that $[\hat{L}_z, U] = 0$, note that

$\bar{U} = \hat{R}_z^\dagger U\hat{R}_z = U(r)$ $(\bar{r} = r)$

& then: $[U,\hat{R}_z] = 0$ ($\hat{R}_z$ is unitary).

For the infinitesimal: $\hat{R}_z \approx (1 - i/\hbar\,\theta\hat{L}_z)$, then

$$[U, 1 - i/\hbar\,\theta\hat{L}_z] = 0 = \cancel{[U,1]} - i/\hbar\,\theta[U,\hat{L}_z]$$

& $[U,\hat{L}_z] = 0 \Rightarrow [\hat{L}_z,\hat{H}] = \frac{1}{2m}\underbrace{[\hat{L}_z,\hat{p}^2]}_{\text{(last time)}} + \underbrace{[\hat{L}_z,U]}_{0\ \text{(this time)}} = 0$

$\hat{L}_z$ & $\hat{H}$ commute, so we can find simultaneous eigenstates of both.

Sim. $[\hat{L}^2,\hat{H}] = 0$, & the Hydrogenic $|n\ell m\rangle$ are eigenstates of: $\hat{H}$, $\hat{L}_z$ & $\hat{L}^2$.