Noether's Theorem in QM.

The infinitesimal form of Noether's theorem:
Given a transformation operator $\hat{Q}$, w/ infinitesimal generator $\hat{q}$ (Hermitian)

$$\hat{\hat{H}} = \hat{Q}^{+}\hat{H}\hat{Q} = (1+i\epsilon\hat{q})\hat{H}(1-i\epsilon\hat{q})$$

$$\overset{!}{=} \hat{H} + i\epsilon(\hat{q}\hat{H} - \hat{H}\hat{q}) + O(\epsilon^2)$$

$$\overset{!}{=} \hat{H} + i\epsilon[\hat{q},\hat{H}]$$

+ $\frac{d\langle\hat{q}\rangle}{dt} = \frac{i}{\hbar}\langle[\hat{H},\hat{q}]\rangle$ so that if $\hat{\hat{H}} = \hat{H}$ (symmetry), $[\hat{q},\hat{H}]=0$,

+ $$\frac{d\langle\hat{q}\rangle}{dt} = \frac{i}{\hbar}\langle[\hat{H},\hat{q}]\rangle = 0 \checkmark$$

conservation.

But there's a larger (non-infinitesimal) setting:

$$\hat{\hat{H}} = \hat{Q}^{+}\hat{H}\hat{Q} = \hat{H} \quad \Rightarrow [\hat{H},\hat{Q}] = 0$$

symmetry

leads to $\frac{d\langle\hat{Q}\rangle}{dt} = \frac{i}{\hbar}\langle[\hat{H},\hat{Q}]\rangle \overset{!}{=} 0$

conservation

In QM, then, both the transformation $\hat{Q}$ + its infinitesimal generator, $\hat{q}$, satisfy Noether's theorem.

That leads to some interesting opportunities not available in CM.

example: $\hat{T} \doteq (1 - i\epsilon/\hbar\,\hat{p})$ for infinitesimal generator $\hat{p}$.

Given $\hat{H} = \hat{p}^2/2m + U(\hat{x})$,

$$[\hat{p},\hat{H}] = [\hat{p},U] = 0 \text{ means:}$$

+ $$[\hat{p},U]f(x) = \frac{\hbar}{i}\frac{d}{dx}(Uf) - U\frac{\hbar}{i}\frac{df}{dx} = \frac{\hbar}{i}\frac{dU}{dx} = 0$$

i.e., no non-trivial potentials (a uniform potential induces an undetectable energy shift).

So for $[\hat{p},\hat{H}] = 0 \Rightarrow \frac{d\langle\hat{p}\rangle}{dt} = 0$, we have no force on the particle. ✓

How about the CM $\hat{T}(a)$ operator?

$[\hat{T},\hat{H}] = [\hat{T},U] = 0$, now we have:

$$[\hat{T},U]f(x) = \hat{T}(Uf) - U\hat{T}f$$

$$\overset{!}{=} U(x-a)f(x-a) - U(x)f(x-a) = 0$$

this can happen if $U(x-a) = U(x)$ for a particular value of a – a periodic potential has this property.

Bloch's Theorem

Suppose we have a periodic potential: $U(x-a) = U(x)$
then: $[\hat{T},\hat{H}] = 0$, + we can form simultaneous eigenstates of $\hat{H}$:

$$\hat{H}\psi(x) = E\psi(x)$$

+ $\hat{T}$: $$\hat{T}\psi(x) = \lambda\psi(x)$$

$\hat{T}$ is a unitary operator: $\hat{T}^{+} = \hat{T}^{-1}$, & these have $|\lambda| = \pm 1$:

$$\hat{T}|\psi\rangle = \lambda|\psi\rangle \quad w/ \quad \langle\psi|\hat{T}^{+} = \lambda^{*}\langle\psi|$$

so that
$$\langle\psi|\underbrace{\hat{T}^{+}\hat{T}}_{=1}|\psi\rangle = \lambda\lambda^{*}\langle\psi|\psi\rangle$$
$$\langle\psi|\psi\rangle = \lambda\lambda^{*}\langle\psi|\psi\rangle \Rightarrow \lambda\lambda^{*} = 1 \checkmark$$

In general, $\lambda = e^{i\phi}$ ← real, &

$$\hat{T}\psi(x) = \lambda\psi(x) \longrightarrow \psi(x-a) = e^{i\phi}\psi(x)$$

i.e. translation induces a phase shift. Equivalently:

$$\psi(x) = e^{i\sigma x}u(x) \quad \text{for a function } u(x) \text{ w/ } u(x-a) = u(x).$$

then
$$\psi(x-a) = e^{i\sigma(x-a)}u(x-a)$$
$$\overset{!}{=} e^{-i\sigma a}\cdot e^{i\sigma x}u(x) = e^{-i\sigma a}\psi(x) \checkmark$$
$$(\phi \equiv -\sigma a).$$

Parity Operator

The operator $\hat{\pi}$ acts on $\psi(x)$ via: $\hat{\pi}\psi(x) = \psi(-x)$

note that
$$\hat{\pi}\hat{\pi}\psi(x) = \psi(x), \text{ so } \hat{\pi}^{-1} = \hat{\pi}. \ (\text{also } \hat{\pi}^{+} = \hat{\pi})$$

As w/ translation, we can define the action of $\hat{\pi}$ on an operator $\hat{Q}$:

$$\hat{\hat{Q}} = \hat{\pi}^{+}\hat{Q}\hat{\pi}$$

example: $\hat{\hat{x}} = \hat{\pi}^{+}\hat{x}\hat{\pi}$ has

$$\hat{\hat{x}}f(x) = \hat{\pi}^{+}\hat{x}\hat{\pi}f(x)$$
$$\overset{!}{=} \hat{\pi}^{+}\hat{x}f(-x) = \hat{\pi}^{+}(xf(-x)) = -xf(x)$$

so $\hat{\hat{x}} = -\hat{x}$, & similarly,

$$\hat{\hat{p}}f(x) = \hat{\pi}^{+}\hat{p}\hat{\pi}f(x) = \hat{\pi}^{+}\frac{\hbar}{i}\frac{d}{dx}f(-x) = \hat{\pi}^{+}\left(-\frac{\hbar}{i}\frac{df}{dx}\Big|_{x=-x}\right)$$
$$\overset{!}{=} -\hat{p}f(x) \quad \text{so} \quad \hat{\hat{p}} = -\hat{p}.$$

& for a general operator $\hat{Q}(\hat{x},\hat{p})$, we have:

$$\hat{\hat{Q}} = \hat{Q}(-\hat{x},-\hat{p})$$

In 3 dimensions, $\hat{\pi}$ acts on $\psi(\vec{r})$ via: $\hat{\pi}\psi(\vec{r}) = \psi(-\vec{r})$ & everything carries over:

$$\hat{\hat{\vec{r}}} = -\hat{\vec{r}}, \quad \hat{\hat{\vec{p}}} = -\hat{\vec{p}}, \quad \hat{\hat{Q}} = \hat{Q}(-\vec{r},-\vec{p})$$

consider the response of the operator: $\hat{\vec{L}} = \hat{\vec{r}}\times\hat{\vec{p}}$:

$$\hat{\hat{\vec{L}}} = (-\hat{\vec{r}})\times(-\hat{\vec{p}}) = \hat{\vec{r}}\times\hat{\vec{p}} = \hat{\vec{L}}$$

The response of the vector operator $\hat{\vec{L}}$ is different from the vector operators $\hat{\vec{r}}$ & $\hat{\vec{p}}$.

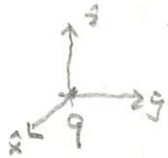
vector op.s that respond like $\hat{\vec{r}}$ (& $\hat{\vec{p}}$) are "true" vector op.s
" " " " " $\hat{\vec{L}}$ are "pseudovector" op.s.

Vectors & Pseudovectors

The distinction exists outside of QM (although it is not so useful).

The electric field $\vec{E}(\vec{r})$ is a "true vector": $\vec{E}(-\vec{r}) = -\vec{E}(\vec{r})$

example:



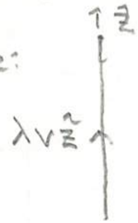
$$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2} = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3}$$

has

$$\vec{E}(-\vec{r}) = \frac{-q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3} = -\vec{E}(\vec{r}) \checkmark$$

The magnetic field $\vec{B}(\vec{r})$ is a pseudovector: $\vec{B}(-\vec{r}) = \vec{B}(\vec{r})$

example:



$$\vec{B}(\vec{r}) = \frac{\mu_0 \lambda v}{2\pi s}\hat{\phi} = \frac{\mu_0 \lambda v}{2\pi s}(-y\hat{x} + x\hat{y})$$

w/

$$\vec{B}(-\vec{r}) = \frac{\mu_0 \lambda (-v)}{2\pi s}(y\hat{x} - x\hat{y}) = \frac{\mu_0 \lambda v}{2\pi s}(-y\hat{x} + x\hat{y}) = \vec{B}(\vec{r})$$

Hydrogenic States

To see the implications of the response to $\hat{\Pi}$ & other 3-dim. operators, we'll use the eigenstates of Hydrogen $|n\ell m\rangle$ w/:

$$\hat{H}|n\ell m\rangle = E_n|n\ell m\rangle$$

$$\hat{L}^2|n\ell m\rangle = \hbar^2 \ell(\ell+1)|n\ell m\rangle$$

$$\hat{L}_z|n\ell m\rangle = \hbar m|n\ell m\rangle$$

for a given n, we have $\ell = 0 \to n-1$ & $m = -\ell \to \ell$

Using the coordinate basis, we have:

$$\langle\theta\phi|n\ell m\rangle \sim e^{im\phi} P_\ell^m(\cos\theta)$$

associated Legendre polynomial

under $x \to -x$, $y \to -y$, $z \to -z$, we have

$$r \to r, \quad \theta \to \pi - \theta \quad \phi \to \pi + \phi$$

&

$$e^{im\phi} P_\ell^m(\cos\theta) \longrightarrow \underbrace{e^{im(\pi+\phi)}}_{(-1)^m e^{im\phi}} P_\ell^m(\underbrace{\cos(\pi-\theta)}_{=-\cos\theta})$$

& $P_\ell^m(-\cos\theta) = (-1)^{\ell+m}$

so $\psi_{n\ell m}(r,\theta,\phi) \longrightarrow \psi_{n\ell m}(r,\pi-\theta,\pi+\phi) = (-1)^\ell \psi_{n\ell m}(r,\theta,\phi)$

or

$$\hat{\Pi}|n\ell m\rangle = (-1)^\ell |n\ell m\rangle$$

& these are eigenstates of $\hat{\Pi}$ in addition to $\hat{H}$...

That implies $[\hat{\Pi}, \hat{H}] = 0$ - let's check:

$$\hat{H} = \frac{\hat{p}^2}{2m} + U(r) \leftarrow = \frac{-q^2}{4\pi\epsilon_0 r}$$

$$[\hat{\Pi}, \hat{H}] = \frac{1}{2m}[\hat{\Pi}, \hat{p}^2] + [\hat{\Pi}, U]$$

we know that $\hat{p}^2 = \hat{\Pi}^\dagger \hat{p}^2 \hat{\Pi} = (-\hat{p})\cdot(-\hat{p}) = \hat{p}^2$

so $[\hat{p}^2, \hat{\Pi}] = 0 \checkmark$

and $\bar{U} = \hat{\Pi}^\dagger U \hat{\Pi} = U(r) \Rightarrow [\hat{\Pi}, U] = 0$

so

$$[\hat{\Pi}, \hat{H}] = 0 \checkmark$$