Magnetism in Classical Mechanics

The Lagrangian associated w/ a magnetic field $\vec{B}$ is

$$L = \tfrac{1}{2} m v^2 + q \vec{v} \cdot \vec{A} \qquad \text{w/ } \vec{A} \text{ the vector pot'l:} \quad \vec{B} = \nabla \times \vec{A}$$

the Euler-Lagrange eqn.s return:

$$m \ddot{\vec{r}} = q \vec{v} \times (\nabla \times \vec{A}) \quad \checkmark$$

How about the Hamiltonian?

The canonical momenta are: $p_i = \frac{\partial L}{\partial \dot{x}^i}$ $(i=1,2,3)$

w/

$$p_i = m v_i + q A_i \Rightarrow \vec{p} = m\vec{v} + q\vec{A} \Rightarrow \vec{v} = \tfrac{1}{m}(\vec{p} - q\vec{A})$$

then:

$$H = (\vec{p}\cdot\vec{v} - L)\big|_{\vec{p}} = \tfrac{1}{m}(\vec{p}-q\vec{A})\cdot\vec{p} - \tfrac{1}{2m}(\vec{p}-q\vec{A})\cdot(\vec{p}-q\vec{A})$$

combine

$$- \tfrac{1}{m}(\vec{p}-q\vec{A})\cdot(q\vec{A})$$

$$= \tfrac{1}{2m}(\vec{p}-q\vec{A})\cdot(\vec{p}-q\vec{A})$$

the canonical momentum $\vec{p} = m\vec{v} + q\vec{A}$ – the "mechanical momentum" is $\vec{p}_m \equiv m\vec{v}$, so $\vec{p} = \vec{p}_m + q\vec{A}$, →

$H = \frac{1}{2m} \vec{p}_m \cdot \vec{p}_m = \frac{1}{2} m v^2$, as expected for magnetism.

? The quantum prescription is: "momentum $\rightarrow \frac{\hbar}{i}\nabla$" but which momentum?

Magnetism in Quantum Mechanics

using $\vec{p} \rightarrow \frac{\hbar}{i}\nabla$, $E \rightarrow i\hbar \frac{\partial}{\partial t}$,

$$E = \tfrac{1}{2m}(\vec{p}-q\vec{A})\cdot(\vec{p}-q\vec{A}) \longrightarrow i\hbar \frac{\partial\psi}{\partial t} = \tfrac{1}{2m}\left[\left(\tfrac{\hbar}{i}\nabla - q\vec{A}\right)\cdot\left(\tfrac{\hbar}{i}\nabla - q\vec{A}\right)\right]\psi$$

or, in its t-indep. form:

$$\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - q\vec{A}\right)\cdot\left(\frac{\hbar}{i}\nabla - q\vec{A}\right)\psi = E\psi.$$

issue: $\vec{A} \rightarrow \vec{A} + \nabla g$ for scalar function g
has $\nabla \times (\vec{A} + \nabla g) = \nabla \times \vec{A} = \vec{B}$ (gauge freedom.)

in CM, the field $\vec{B}$ shows up in the eqn.s of motion – but in QM, $\vec{A}$ is all we get!

What happens, physically, when we take $\vec{A} \rightarrow \vec{A} + \nabla g$?

Is $\vec{A}$ by itself detectable?

"Aharonov – Bohm"

Particle constrained to move in a circle of radius R:

$$-\frac{\hbar^2}{2m}\cdot\frac{1}{R^2}\frac{d^2\psi(\phi)}{d\phi^2} = E\,\psi(\phi)$$

w/ $\psi(\phi + 2\pi) = \psi(\phi)$ the "boundary condition"

$$\psi = A e^{i\sqrt{\frac{2mR^2}{\hbar^2}E}\,\phi} + B e^{-i\sqrt{\frac{2mR^2}{\hbar^2}E}\,\phi}$$

(what type of solutions are these?)

For either solution, the periodicity condition requires:

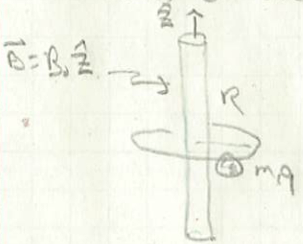
$$e^{\pm i\sqrt{\frac{2mR^2}{\hbar^2}E}\,(\phi+2\pi)} = e^{\pm i\sqrt{\frac{2mR^2}{\hbar^2}E}\,\phi} \cdot \underbrace{e^{\pm i\sqrt{\frac{2mR^2}{\hbar^2}E}\cdot 2\pi}}_{=1}$$

so

$$\sqrt{\frac{2mR^2}{\hbar^2}E} = n \Rightarrow E = \frac{n^2\hbar^2}{2mR^2}$$

energy is quantized, w/ the <u>same</u> energy for left & right travellers.

Now suppose we put an infinite solenoid through the center of the ring:



What happens classically? nothing, no $\vec{B}$ out there at the particle location.

Quantum mechanics, though, depends on $\vec{A}$, & that is non-zero at the particle: $\vec{A} = A(s)\,\hat{\phi}$

Schrödinger's eqn: $\frac{1}{2m}(\vec{p} - q\vec{A})\cdot(\vec{p} - q\vec{A})\,\psi = E\psi$, $\vec{p} \to \frac{\hbar}{i}\nabla$

expand it out:

$$\frac{1}{2m}\left[-\hbar^2\nabla^2\psi - \frac{\hbar}{i}\nabla\cdot(q\vec{A}\psi) - \frac{\hbar}{i}q\vec{A}\cdot\nabla\psi + q^2A^2\psi\right] = E\psi$$

w/ $\nabla\cdot(q\vec{A}\psi) = q[\psi\underbrace{\nabla\cdot\vec{A}}_{=0 \text{ in Coulomb gauge}} + \vec{A}\cdot\nabla\psi]$

So we have:

$$\frac{1}{2m}\left[-\hbar^2\nabla^2\psi - \frac{\hbar}{i}\,2q\vec{A}\cdot\nabla\psi + q^2A^2\psi\right] = E\psi \qquad (*)$$

assume that ψ has the same periodic form as in the original (no magnetic field) case:

$$\psi = e^{\pm in\phi}$$

then $\frac{\hbar}{i}\nabla\psi = \frac{\hbar}{i}\cdot\frac{1}{R}\frac{\partial\psi}{\partial\phi}\,\hat{\phi} = \pm\frac{\hbar n}{R}\psi\hat{\phi}$

$$-\hbar^2\nabla^2\psi = -\hbar^2\frac{1}{R^2}\frac{\partial^2\psi}{\partial\phi^2} = \frac{\hbar^2n^2}{R^2}\psi$$

& (*) reads:

$$\frac{\hbar^2}{2mR^2}\left[n^2 \mp \frac{2nRq\overset{A(R)}{A}}{\hbar} + q^2\frac{\overset{A(R)^2}{A^2}R^2}{\hbar^2}\right]\psi = E\psi$$

or

$$\frac{\hbar^2}{2mR^2}\left(\pm n - \frac{qAR}{\hbar}\right)^2\psi = E\psi$$

so the energies are now:

$$E = \frac{\hbar^2}{2mR^2}\left(\pm n - \frac{qAR}{\hbar}\right)^2 \qquad (+)$$

& the energy of the left & right traveller's is <u>different</u>, detectably different.

Back to the question of gauge – does this result depend on $\vec{A}$ vs. $\vec{A} + \nabla g$?

From $\vec{B} = \nabla\times\vec{A}$, integrate over a disk of radius R:

$$\underbrace{\int_D \vec{B}\cdot d\vec{a}}_{\text{magnetic flux through the disk - gauge invariant} \;=\; \Phi} = \underbrace{\int (\nabla\times\vec{A})\cdot d\vec{a}}_{\text{curl theorem}} = \int_{\partial D} \vec{A}\cdot d\vec{l} = A(R)\cdot 2\pi R$$

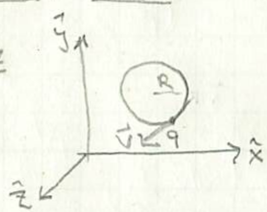
so $A(R) = \dfrac{\Phi}{2\pi R}$, a gauge independent quantity.

& the energy result in (+) is gauge-independent.

Motion in a Uniform Magnetic Field

classical story: for $\vec{B} = +B_0\hat{z}$

a charge q undergoes uniform circular motion w/:



$$\frac{mv^2}{R} = qvB_0 \Rightarrow R = \frac{mv}{qB_0} \quad \text{the "cyclotron radius"}$$

&

$$vT = 2\pi R \Rightarrow T = \frac{2\pi R}{v} = \frac{2\pi m}{qB_0} \quad \text{the period}$$

& $\omega \equiv \dfrac{2\pi}{T} = \dfrac{qB_0}{m}$ the frequency.

the particle's energy is: $E = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 R^2$

On the quantum mechanical side, we need $\vec{A}$ w/ $\nabla\times\vec{A} = B_0\hat{z}$

$(\nabla\times\vec{A})_z = \dfrac{\partial A^y}{\partial x} - \dfrac{\partial A^x}{\partial y}$ take $\vec{A} = B_0 x\hat{y}$ (w/ $\nabla\cdot\vec{A} = 0$ ✓)

then the $\frac{1}{2m}q^2A^2\psi$ term in (*) looks like

$$\frac{1}{2m}q^2A^2\psi = \frac{1}{2m}\underbrace{\frac{q^2B_0^2}{m^2}}_{\omega^2}x^2 m^2\psi = \frac{1}{2}m\omega^2x^2\psi$$

a harmonic oscillator potential energy, & the quantized energies here are the same as that case:

$$E_n = (n + \tfrac{1}{2})\hbar\omega$$