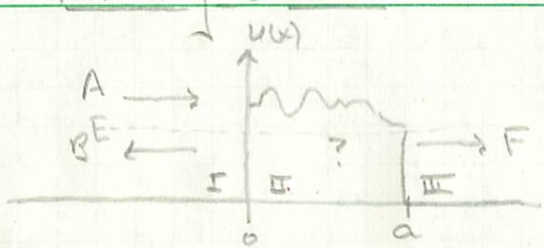
Tunneling & WKB



$p(x) = \sqrt{2m(E-U(x))}$

In region II: $\psi_{II} = \frac{1}{\sqrt{p(x)}}\left[J e^{i\phi(x)/\hbar} + K e^{-i\phi(x)/\hbar}\right]$

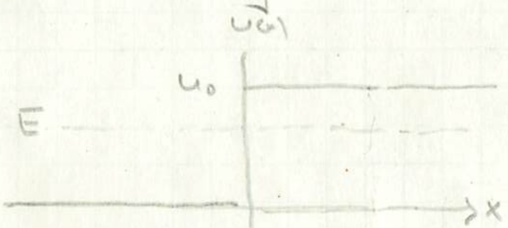
& $\phi(x) = \int_0^x p(\bar{x})\,d\bar{x} = \int_0^x \sqrt{\underbrace{2m(E-U(\bar{x}))}_{<0 \text{ in II}}}\,d\bar{x}$

$= i\int_0^x |p(\bar{x})|\,d\bar{x} = i|\phi(x)|$

& $\psi_{II} = \frac{1}{\sqrt{p(x)}}\left[J e^{-|\phi(x)|/\hbar} + K e^{|\phi(x)|/\hbar}\right]$ (*)

growing & decaying exponentials.

Aside – tunneling: Consider the simplified U(x):



inside the step,

$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U_0\psi = E\psi$

so $\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}\underbrace{(U_0-E)}_{>0}\psi \Rightarrow \psi = J e^{-\sqrt{2m(U_0-E)}\,x/\hbar} + K e^{\sqrt{2m(U_0-E)}\,x/\hbar}$

exactly what we get from WKB.

Think about the normalization condition – w/ the growing exponential in place, we have:

$|\psi|^2 \sim |K|^2 e^{2\sqrt{2m(U_0-E)}\,x/\hbar}$

this term dominates.

& $\int_0^x |\psi(\bar{x})|^2 d\bar{x} = \frac{|K|^2\hbar}{2\sqrt{2m(U_0-E)}}\left[e^{2\sqrt{2m(U_0-E)}\,x/\hbar} - 1\right]$

let $2\sqrt{2m(U_0-E)}\cdot\frac{1}{\hbar} \equiv Q$

$\simeq \frac{|K|^2}{Q} e^{Qx}$

the probability grows exponentially, & suppose that for $x=\ell$,

$\frac{|K|^2}{Q} e^{Q\ell} = 1 \Rightarrow |K|^2 = \frac{Q}{e^{Q\ell}}$

if $Q \gg 1 \Rightarrow U_0 \gg E$ or $Q\ell \gg 1$,

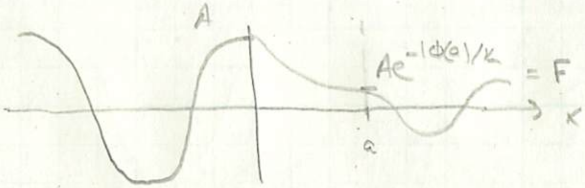
$|K|^2 \longrightarrow 0$, we lose the growing exponential

leaving $\psi = J e^{-\sqrt{2m(U_0-E)}\,x/\hbar}$, the decaying (tunneling) form.

The same argument holds for the term in (*), so

$\psi_{II} = \frac{1}{\sqrt{p(x)}} J e^{-|\phi(x)|/\hbar}$

Cartoon sketch of (the real part of) ψ:



so $F \approx Ae^{-|\phi(a)|/\hbar}$,

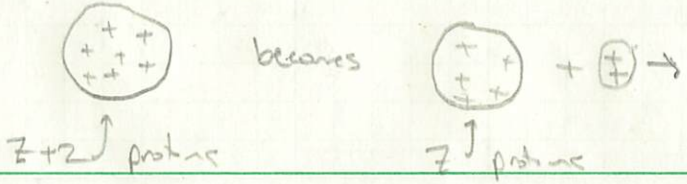
$$T = \frac{|F|^2}{|A|^2} = e^{-2|\phi(a)|/\hbar} \quad \text{w/} \quad \phi(x) = \int_0^x \sqrt{2m(E-U(\bar{x}))}\,d\bar{x}$$

Alpha Decay

Observation – certain atoms emit alpha particles (2n+2p) w/ energy E – these atoms "decay" over time – what is the relation between the energy E & the "lifetime" of the atom?

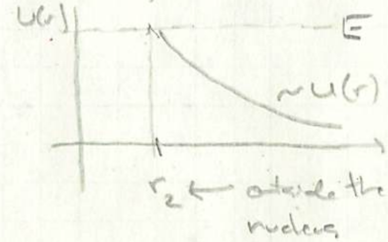
Question 1 – what is the mechanism by which this happens?

the strong force holds the nucleus of atoms together. If a particle w/ charge Ze could escape the nucleus, it would be repelled by the other protons in the nucleus, & ejected from the atom:



& once outside the nucleus, the alpha encounters a potential:

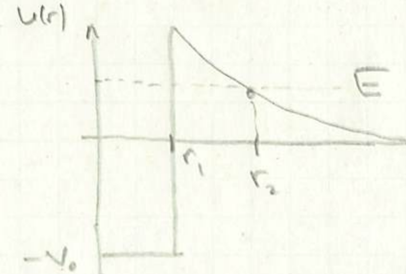
$$U(r) = \frac{Ze^2}{4\pi\epsilon_0 r}$$



suppose the particle energy is E, &

$$E = U(r_2) = \frac{Ze^2}{4\pi\epsilon_0 r_2} \Rightarrow r_2 = \frac{Ze^2}{4\pi\epsilon_0 E}.$$

but there is some "potential" that keeps the protons inside the nucleus – Gamow modelled it as a well:



the problem is – classically, no way for the particle to "escape" from r_1 to $> r_2$ at energy E.

in QM, there is tunneling. The probability that a particle tunnels from $r_1 \to r_2$ is:

$$T \approx e^{-2\phi(r_2)/\hbar} \quad \text{w/} \quad \phi(r) = \int_{r_1}^{r} \sqrt{2m(U(\bar{r})-E)}\,d\bar{r} \quad \text{(WKB)}$$

Semi-classical – if the particle runs up against r_1 every $\bar{\tau}$ seconds, then the probability of escape, per unit time, is

$$R = \frac{e^{-2\phi(r_2)/\hbar}}{\bar{\tau}} \quad \text{(rate)}$$

& the "lifetime" of the particle, $\tau = 1/R$

$$\tau = \hat{t} \cdot e^{2\Phi(r_2)/\hbar}$$

to find the dominant contribution to the lifetime, we need to evaluate:

$$\Phi(r) = \sqrt{2m} \int_{r_1}^{r} \sqrt{\frac{Ze^2}{4\pi\epsilon_0 \bar{r}} - E} \, d\bar{r}$$

$$= \sqrt{2mE} \int_{r_1}^{r} \sqrt{\frac{r_2}{\bar{r}} - 1} \, d\bar{r}$$

&

$$\Phi(r_2) = \sqrt{2mE} \int_{r_1}^{r_2} \sqrt{\frac{r_2 - \bar{r}}{\bar{r}}} \, d\bar{r} \qquad \text{let } \bar{r} = r_2 \sin^2(u)$$

$$= \sqrt{2mE} \int_{\sin^{-1}(r_1/r_2)}^{\sin^{-1}(1)} \left[\frac{\cos^2 u}{\sin^2 u}\right]^{1/2} 2r_2 \sin(u)\cos(u)\, du$$

$$= 2r_2\sqrt{2mE} \int_{\sin^{-1}(r_1/r_2)}^{\sin^{-1}(1)} \cos^2 u \, du$$

$$= 2r_2\sqrt{2mE} \int_{\sin^{-1}(r_1/r_2)}^{\sin^{-1}(1)} (1 - \sin^2 u)\, du$$

dominant term

& $\sin^{-1}(1) = \pi/2$,

$$\Phi(r_2) \approx 2r_2\sqrt{2mE}\left[\pi/2 - \sin^{-1}(r_1/r_2)\right]$$

dominant term

$$\simeq \frac{\pi\sqrt{2mE}}{4\pi\epsilon_0 E} \frac{1}{Ze^2} = K \frac{1}{\sqrt{E}}$$

the theory predicts that

$$\tau = \hat{t} \cdot {}^{\hat{K}/\sqrt{E}} \qquad \text{w/ } \hat{K} \equiv 2K/\hbar$$

this scaling of $\tau \sim e^{\hat{K}/\sqrt{E}}$ matches the observations!