Problem Set 2

Physics 367
Computational Methods for Physics

Due on Friday, September 11th, 2026

Problem 1

Suppose you have the (vector) differential equation:

$$\frac{d\mathbf{f}(t)}{dt} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}}_{\equiv \mathbb{A}} \mathbf{f}(t), \tag{1}$$

with initial condition $\mathbf{f}(0) = \mathbf{f}_0$. If you discretize in time, letting $t_j = j\Delta t$ for fixed Δt, then the approximation to the derivative is:

$$\frac{\mathbf{f}(t_{j+1}) - \mathbf{f}(t_j)}{\Delta t} = \mathbb{A}\mathbf{f}(t_j). \tag{2}$$

a. Write the "update" relating $\mathbf{f}(t_{j+1})$ to $\mathbf{f}(t_j)$ in the form: $\mathbf{f}(t_{j+1}) = \mathbb{M}\mathbf{f}(t_j)$ ($\mathbb{M}$ is a matrix) and compute the absolute value of the eigenvalues of $\mathbb{M}$.

b. You can define a related update by starting with:

$$\frac{\mathbf{f}(t_{j+1}) - \mathbf{f}(t_j)}{\Delta t} = \mathbb{A}\mathbf{f}(t_{j+1}). \tag{3}$$

Find the matrix $\mathbb{P}$ in the new update $\mathbf{f}(t_{j+1}) = \mathbb{P}\mathbf{f}(t_j)$ and compute the absolute value of the eigenvalues of $\mathbb{P}$.

c. Can you come up with a linear combination of these two approaches that yields an "update" matrix that has eigenvalues with unit magnitude? i.e. starting from:

$$\frac{\mathbf{f}(t_{j+1}) - \mathbf{f}(t_j)}{\Delta t} = \mathbb{A}\left(\alpha\mathbf{f}(t_j) + (1-\alpha)\mathbf{f}(t_{j+1})\right), \tag{4}$$

can you pick α such that the matrix appearing in $\mathbf{f}(t_{j+1}) = \mathbb{Q}\mathbf{f}(t_j)$ has eigenvalues with $|\lambda| = 1$?

Problem 2

For a symmetric matrix, $\mathbb{A} \in \mathbf{R}^{n\times n}$, with eigenvalues that have the property: $\lambda_1 > \lambda_2 > \ldots > \lambda_n > 1$, show that for a random (meaning here, one that contains a mixture of all of the eigenvectors of the matrix) non-zero vector, $\mathbf{x} \in \mathbf{R}^n$,

$$\mathbb{A}^p \mathbf{x} \parallel \mathbf{v}_1, \tag{5}$$

as $p \to \infty$, where $\mathbf{v}_1$ is the eigenvector associated with the largest eigenvalue.

Problem 3

Taylor expand the position vector, $\mathbf{r}(t_j + \Delta t)$, in small Δt through Δt^3. Rewrite the resulting expression in terms of $\mathbf{v}(t) \equiv \dot{\mathbf{r}}(t)$ and $\mathbf{a}(t) = \mathbf{F}(\mathbf{r}(t))/m$ (we're assuming Newton's second law is satisfied), and compare with the Verlet position update from class: $\bar{\mathbf{r}}_{j+1} = \bar{\mathbf{r}}_j + \bar{\mathbf{v}}_j \Delta t + (1/2)\Delta t^2 \mathbf{F}(\bar{\mathbf{r}}_j)/m$. What do you conclude about the error, per step, of the method (at least as far as the position is concerned)? Do the same thing for $\mathbf{v}(t_j + \Delta t)$ and again compare with the update: $\bar{\mathbf{v}}_{j+1} = \bar{\mathbf{v}}_j + (1/2)\Delta t(\mathbf{F}(\bar{\mathbf{r}}_j) + \mathbf{F}(\bar{\mathbf{r}}_{j+1}))/m$ – what is the error per step in velocity?

Problem 4

Write Euler's equations governing the spin components of a rigid body (with no net applied torque):

$$\begin{aligned} 0 &= (I_2 - I_3)\omega_2\omega_3 - I_1\dot{\omega}_1 \\ 0 &= (I_3 - I_1)\omega_3\omega_1 - I_2\dot{\omega}_2 \\ 0 &= (I_1 - I_2)\omega_1\omega_2 - I_3\dot{\omega}_3 \end{aligned} \tag{6}$$

(where the constants I_1, I_2 and I_3 describe the shape of the body) as a first order differential equation of the form: $\frac{d\mathbf{f}}{dt} = \mathbf{G}(\mathbf{f})$ to do this, start by defining every component of $\mathbf{f}$ (for example, take $f_1 = \omega_1$, $f_2 = \omega_2$, $f_3 = \omega_3$) and write out the function $\mathbf{G}$ explicitly in terms of the components $\{f_j\}_{j=1}^n$ (for n appropriate to this problem).

Problem 5

Write the harmonic oscillator equation of motion: $\ddot{x}(t) = -\omega^2 x(t)$ in first order form by defining the vectors $\mathbf{f}$ and $\mathbf{G}$ in $\frac{d\mathbf{f}}{dt} = \mathbf{G}$. Note that the problem is linear, so you can write $\mathbf{G} = \mathbb{A}\mathbf{f}$; identify the components of the matrix $\mathbb{A}$. Now, given the initial conditions, $x(0) = x_0$ and $\dot{x}(0) = v_0$, compute the solution to

$$\frac{d\mathbf{f}(t)}{dt} = \mathbb{A}\mathbf{f}(t) \text{ with } \mathbf{f}(0) = \mathbf{f}_0, \tag{7}$$

using matrix exponentiation (you can get a symbolic algebra package to compute $e^{\mathbb{A}t}$).