Problem 1 | | Problem Set 9

For $\vec{E} = E_0 \cos(k(z-ct))\hat{y}$, we have: $x(t)=0$, $z(t)=0$ (for a particle that starts at rest at the origin), so

$$\ddot{y}(t) = \frac{qE_0}{m}\cos(\omega t) \qquad \text{w/ } \omega \equiv kc.$$

$t_r \equiv t - r/c$

$$\text{so} \quad \vec{B}_d = \frac{-\mu_0}{4\pi rc}\hat{r}\times\ddot{\vec{p}} = -\frac{\mu_0 q^2 E_0}{4\pi m}\frac{\cos(\omega t_r)}{rc}\hat{r}\times\left[\sin\theta\sin\phi\,\hat{r} + \cos\theta\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi}\right]$$

$$= -\frac{\ell}{r}\frac{E_0}{c}\cos(\omega t_r)\left[\cos\theta\sin\phi\,\hat{\phi} - \cos\phi\,\hat{\theta}\right] \quad \text{w/ } \ell \equiv \frac{\mu_0 q^2}{4\pi m}$$

$$\text{so} \quad \vec{E}_d = -c\hat{r}\times\vec{B}_d = E_0\, \ell/r\, \cos(\omega t_r)\left[-\cos\theta\sin\phi\,\hat{\theta} - \cos\phi\,\hat{\phi}\right]$$

$$\text{Then} \quad \vec{S}_d = \frac{1}{\mu_0}\vec{E}_d\times\vec{B}_d = \frac{E_0^2}{\mu_0 c}\frac{\ell^2}{r^2}\cos^2(\omega t_r)\left[\cos^2\theta\sin^2\phi\,\hat{r} + \cos^2\phi\,\hat{r}\right]$$

Time-averaging gives: $\vec{I}_d = \langle\vec{S}_d\rangle = I_0\frac{\ell^2}{r^2}\left[\cos^2\theta\sin^2\phi + \cos^2\phi\right]\hat{r}$ No ω dep. here since the amplitude of oscillation itself goes like $\sim 1/\omega^2$, then the acceleration has $\omega^2/\omega^2 \to 1$ larger ω leads to smaller amp. unlike a dipole I drive: $p = qd\cos(\omega t)$ w/ d & ω unrelated.

Problem 2

a. $$\rho(x,0) = \int_{-\infty}^{+\infty} A(k)\, e^{-i2\pi kx}\, dk = u(x)$$

The i-Fourier-transform of $u(x)$ is: $u(x) = \int_{-\infty}^{+\infty}\tilde{u}(k)\, e^{-i2\pi kx}\, dk$

so $$A(k) = \tilde{u}(k) = \int_{-\infty}^{+\infty} u(x)\, e^{i2\pi kx}\, dx.$$

(i.e. $A(k)$ is the FT of $u(x)$.)

b. For $f(k) = f(k_0 + \Delta k) \approx f(k_0) + f'(k_0)\Delta k$, ($= k - k_0$)

$$f(x,t) = \int_{-\infty}^{+\infty} A(k)\, e^{i2\pi(f(k_0) + f'(k_0)\underbrace{\Delta k}_{=k-k_0})t}\, e^{-i2\pi\lambda x}\, d\lambda \quad (\text{only } \lambda \approx \lambda_0 \text{ contribute})$$

$$= \underbrace{e^{i2\pi f(k_0)t}\, e^{-i2\pi f'(k_0)k_0 t}}\int_{-\infty}^{+\infty}\underbrace{A(k)}_{=\tilde{u}(k)}\, e^{i2\pi(f'(k_0)t - x)k}\, dk$$

$$= e^{i\phi t}\int_{-\infty}^{+\infty}\tilde{u}(k)\, e^{-i2\pi k(x - f'(k_0)t)}\, dk$$

$$= e^{i\phi t}\, u(x - f'(k_0)t)$$

right-travelling waveform w/ speed $f'(k_0)$.

Problem 3

For a spherical shell of charge w/ radius R, $\sigma = \frac{Q}{4\pi R^2}$ . . . +

$$\vec{E} = \begin{cases} 0 & r<R \\ \left(Q/4\pi\epsilon_0 r^2\right)\hat{r} & r>R \end{cases}$$

the total energy stored in the field is:

$$W = 4\pi\int_R^\infty \tfrac{1}{2}\epsilon_0 E^2\, r^2\, dr = 2\pi\epsilon_0\int_R^\infty \frac{Q^2}{16\pi^2\epsilon_0^2 r^2}\,dr = \frac{Q^2}{8\pi\epsilon_0}\left(-\frac{1}{r}\right)\Bigg|_R^\infty = \frac{Q^2}{8\pi\epsilon_0 R}.$$

the classical charge radius R: $W = mc^2 = \frac{Q^2}{8\pi\epsilon_0 R} \Rightarrow R = \frac{Q^2}{8\pi\epsilon_0 c^2 m} = \frac{\mu_0 Q^2}{8\pi m}$

For a homogeneous ball of charge w/ $\rho = Q/(4/3\pi R^3)$,

$$\vec{E} = \begin{cases} \frac{\rho r}{3\epsilon_0}\hat{r} & r<R \\ Q/(4\pi\epsilon_0 r^2)\hat{r} & r>R \end{cases}$$

+ the interior solution is $\vec{E}_{in} = \frac{Q r}{4/3\pi R^3 \cdot 3\epsilon_0}\hat{r} = \frac{Qr}{4\pi\epsilon_0 R^3}\hat{r}$

the total work required to build this configuration is:

$$W = \tfrac{1}{2}\epsilon_0\left[4\pi\int_0^R \frac{Q^2 r^4}{16\pi^2\epsilon_0^2 R^6}\,dr + 4\pi\int_R^\infty \frac{Q^2}{16\pi^2\epsilon_0^2 r^2}\,dr\right]$$

$$= \frac{Q^2}{8\pi\epsilon_0 R^6}\frac{1}{5}R^5 + \frac{Q^2}{8\pi\epsilon_0}\left(-\frac{1}{r}\right)\Bigg|_{r=R}^\infty = \frac{Q^2}{8\pi\epsilon_0 R}\left(\frac{1}{5}+1\right) = \frac{3/5\, Q^2}{4\pi\epsilon_0 R}$$

the classical charge radius R: $W = mc^2 = \frac{3/5\,Q^2}{4\pi\epsilon_0 R} \Rightarrow R = \frac{3/5\,Q^2}{4\pi\epsilon_0 c^2 m} = \frac{3/5\,\mu_0 Q^2}{4\pi m}$

Problem 4

For $\vec{E}_{in} = E_{in}\cos(k(z-ct))\hat{n}$, the outgoing radiation has intensity is

$$\vec{S}_{out} = \frac{E_{in}^2}{\mu_0 c}\frac{\ell^2}{r^2}\cos^2(kct_r)\sin^2\alpha\,\hat{r}$$

α: angle between $\hat{n}$ + $\hat{r}$

w/ $\vec{I}_{out} = I_{in}\frac{\ell^2}{r^2}\sin^2\alpha\,\hat{r}$

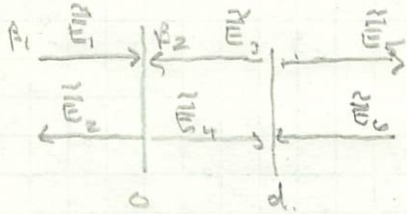
Then $\frac{d\sigma}{d\Omega} = \ell^2\sin^2\alpha = \ell^2\left(1-\sin^2\theta\cos^2(\phi-2\phi)\right)$ w/ polarization average

$$\overline{\frac{d\sigma}{d\Omega}} = \tfrac{1}{2}\ell^2\left(1+\cos^2\theta\right)$$

Problem 5 (9.36)

For the setup:  we had: $\tilde{E}_{10} e^{ik_1 a} + \tilde{E}_{20} e^{-ik_1 a} = \tilde{E}_{30} e^{-ik_2 a} + \tilde{E}_{40} e^{ik_2 a}$

$$\tilde{E}_{10} e^{ik_1 a} - \tilde{E}_{20} e^{-ik_1 a} = \beta\left[-\tilde{E}_{30} e^{-ik_2 a} + \tilde{E}_{40} e^{ik_2 a}\right]$$

We get two copies of these eqn.s for  w/ $\beta_1 = n_2/n_1$, $\beta_2 = n_3/n_2$

At $z=0$, these read:

$$\tilde{E}_{10} + \tilde{E}_{20} = \tilde{E}_{30} + \tilde{E}_{40} \qquad (i)$$

$$\tilde{E}_{10} - \tilde{E}_{20} = \beta_1\left[-\tilde{E}_{30} + \tilde{E}_{40}\right] \qquad (ii)$$

at $z=d$, we have:

$$\tilde{E}_{40} e^{ik_2 d} + \tilde{E}_{30} e^{-ik_2 d} = \tilde{E}_{60} e^{-ik_3 d} + \tilde{E}_{50} e^{ik_3 d} \qquad (iii)$$

$$\tilde{E}_{40} e^{ik_2 d} - \tilde{E}_{30} e^{-ik_2 d} = \beta_2\left[-\tilde{E}_{60} e^{-ik_3 d} + \tilde{E}_{50} e^{ik_3 d}\right] \qquad (iv)$$

The incoming wave on the far right doesn't exist, $\tilde{E}_6 = 0$, & we can write (iii) & (iv) as:

$$\underbrace{\begin{pmatrix} e^{-ik_2 d} & e^{ik_2 d} \\ -e^{-ik_2 d} & e^{ik_2 d} \end{pmatrix}}_{\mathbb{A}} \begin{pmatrix} \tilde{E}_{30} \\ \tilde{E}_{40} \end{pmatrix} = \underbrace{\begin{pmatrix} e^{ik_3 d} & e^{-ik_3 d} \\ \beta_2 e^{ik_3 d} & -\beta_2 e^{-ik_3 d} \end{pmatrix}}_{\mathbb{B}} \begin{pmatrix} \tilde{E}_{50} \\ 0 \end{pmatrix}$$

From (i) & (ii),

$$\underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{\mathbb{C}} \begin{pmatrix} \tilde{E}_{10} \\ \tilde{E}_{20} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ -\beta_1 & \beta_1 \end{pmatrix}}_{\mathbb{D}} \begin{pmatrix} \tilde{E}_{30} \\ \tilde{E}_{40} \end{pmatrix}$$

Then:

$$\begin{pmatrix} \tilde{E}_{10} \\ \tilde{E}_{20} \end{pmatrix} = \mathbb{C}^{-1}\mathbb{D}\mathbb{A}^{-1}\mathbb{B}\begin{pmatrix} \tilde{E}_{50} \\ 0 \end{pmatrix}$$

which gives

$$\tilde{E}_{50} = 4\tilde{E}_{10} e^{id(k_2-k_3)}\left[(1+\beta_1)(1+\beta_2) + (1-\beta_1)(1-\beta_2)e^{2idk_2}\right]^{-1}$$

$$T = \underbrace{\frac{\mu_1 v_1}{\mu_3 v_3}}_{n_3/n_1}\left|\frac{\tilde{E}_{50}}{\tilde{E}_{10}}\right|^2 = \frac{n_3}{n_1}\left[(1+\beta_1)(1+\beta_2) + (1-\beta_1)(1-\beta_2)e^{2idk_2}\right]^{-1}\left[(1+\beta_1)(1+\beta_2) + (1-\beta_1)(1-\beta_2)e^{-2ik_2 d}\right]^{-1}$$

$$= 16\frac{n_3}{n_1}\left[(1+\beta_1)^2(1+\beta_2)^2 + (1-\beta_1^2)(1-\beta_2^2)\,2\underbrace{\cos(2k_2 d)}_{=1-2\sin^2(k_2 d)} + (1-\beta_1)^2(1-\beta_2)^2\right]^{-1}$$

$$= \frac{16 n_3}{n_1}\left[\underbrace{(1+\beta_1)^2(1+\beta_2)^2 + 2(1-\beta_1^2)(1-\beta_2^2) + (1-\beta_1)^2(1-\beta_2)^2}_{=4(1+\beta_1\beta_2)^2} - 4(1-\beta_1^2)(1-\beta_2^2)\sin^2(k_2 d)\right]^{-1}$$

$$= \frac{4n_3}{n_1}\left[(1+\beta_1\beta_2)^2 - (1-\beta_1^2)(1-\beta_2^2)\sin^2(k_2 d)\right]^{-1}$$

$k_2 = \omega/v_2 = \frac{\omega n_2}{c}$

using $\beta_1 = n_2/n_1$, $\beta_2 = n_3/n_2$, $(1+\beta_1\beta_2)^2 = \frac{1}{n_1^2}(n_1+n_3)^2$

& $(1-\beta_1^2)(1-\beta_2^2) = (1-(n_2/n_1)^2)(1-(n_3/n_2)^2) = \frac{1}{n_1^2 n_2^2}(n_1^2-n_2^2)(n_2^2-n_3^2)$

so

$$T = \frac{4n_3}{n_1} \cdot n_1^2 \left[(n_1+n_3)^2 + \frac{1}{n_2^2}(n_1^2-n_2^2)(n_3^2-n_2^2)\sin^2\left(\frac{\omega n_2}{c}d\right)\right]^{-1}$$

$$T^{-1} = \frac{1}{4n_1n_3}\left[(n_1+n_3)^2 + \frac{1}{n_2^2}(n_1^2-n_2^2)(n_3^2-n_2^2)\sin^2\left(\frac{\omega n_2}{c}d\right)\right]. \quad (*) \quad \text{as desired}$$

<u>Problem 6</u> <u>(9.38)</u>

First note that T from $(*)$ is symmetric under $n_1 \leftrightarrow n_3$, so the fish sees us as well as we see the fish.

Referring to the setup from the previous problem, we have:

$n_1 = 4/3$, $n_2 = 3/2$, $n_3 = 1$, so

$$T = 16/3\left[(7/3)^2 + (2/3)^2\left(\frac{-17}{36}\right)(-5/4)\sin^2\left(\frac{3\omega d}{2c}\right)\right]^{-1}$$

$$= 16/3\left[49/9 + \frac{85}{324}\sin^2\left(\frac{3\omega d}{2c}\right)\right]^{-1} = \frac{1728}{1764 + 85\sin^2\left(\frac{3\omega d}{2c}\right)}$$

the <u>max</u> occurs when $\sin^2\left(\frac{3\omega d}{2c}\right) = 0$, w/ $T_{max} = \frac{1728}{1764} \approx .98$

the <u>min</u> occurs when $\sin^2\left(\frac{3\omega d}{2c}\right) = 1$, $T_{min} = \frac{1728}{1849} \approx .934$.