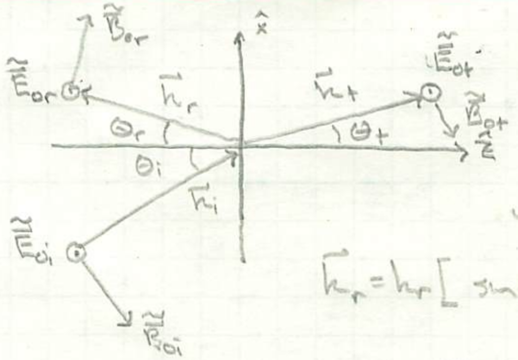
Problem 1 (9.17) | | Problem Set 8



$$\tilde{\vec{E}}_i = \tilde{E}_{oi}\,\hat{y}\,e^{i(\vec{k}_i\cdot\vec{r}-\omega t)}, \quad \tilde{\vec{E}}_r = \tilde{E}_{or}\,\hat{y}\,e^{i(\vec{k}_r\cdot\vec{r}-\omega t)}$$

$$\tilde{\vec{E}}_t = \tilde{E}_{ot}\,\hat{y}\,e^{i(\vec{k}_t\cdot\vec{r}-\omega t)}$$

w/ $\vec{k}_i = k_i[\sin\theta_i\,\hat{x} + \cos\theta_i\,\hat{z}]$

$$\vec{k}_r = k_r[\sin\theta_r\hat{x} - \cos\theta_r\hat{z}], \quad \vec{k}_t = k_t[\sin\theta_t\,\hat{x} + \cos\theta_t\,\hat{z}]$$

The magnetic fields are: $\tilde{\vec{B}}_{oi} = \frac{1}{v_1}\hat{k}_i \times \tilde{\vec{E}}_{oi} = \frac{1}{v_1}\tilde{E}_{oi}[\sin\theta_i\hat{z} - \cos\theta_i\hat{x}]$

$\tilde{\vec{B}}_{or} = \frac{1}{v_1}\hat{k}_r \times \tilde{\vec{E}}_{or} = \frac{1}{v_1}\tilde{E}_{or}[\sin\theta_r\hat{z} + \cos\theta_r\hat{x}]$

$\tilde{\vec{B}}_{ot} = \frac{1}{v_2}\hat{k}_t \times \tilde{\vec{E}}_{ot} = \frac{1}{v_2}\tilde{E}_{ot}[\sin\theta_t\hat{z} - \cos\theta_t\hat{x}]$

there is no discontinuity in the $\hat{z}$ components of $\vec{B}$:

$$\frac{1}{v_1}[\tilde{E}_{oi}\sin\theta_i + \tilde{E}_{or}\sin\theta_r] = \frac{1}{v_2}\tilde{E}_{ot}\sin\theta_t \qquad (1)$$

there is discontinuity in the $\hat{x}$-component:

$$\frac{1}{\mu_1}\cdot\frac{1}{v_1}[-\tilde{E}_{oi}\cos\theta_i + \tilde{E}_{or}\cos\theta_r] = \frac{1}{\mu_2}\cdot\frac{1}{v_2}[-\tilde{E}_{ot}\cos\theta_t] \qquad (2)$$

there is no discontinuity in the $\hat{y}$ component of the electric fields:

$$\tilde{E}_{oi} + \tilde{E}_{or} = \tilde{E}_{ot} \qquad (3)$$

using $\theta_r = \theta_i$ & $\sin\theta_t = \frac{n_1}{n_2}\sin\theta_i$ in (1), (2) & (3):

$$\frac{1}{v_1}\sin\theta_i[\tilde{E}_{oi} + \tilde{E}_{or}] = \frac{1}{v_2}\tilde{E}_{ot}\frac{n_1}{n_2}\sin\theta_i \Rightarrow \tilde{E}_{oi} + \tilde{E}_{or} = \tilde{E}_{ot} \quad (4) \quad \left(1 = \frac{v_1}{v_2}\cdot\frac{n_1}{n_2}\right)$$

$$\tilde{E}_{oi} - \tilde{E}_{or} = \underbrace{\frac{\mu_1 v_1}{\mu_2 v_2}}_{\equiv\beta}\cdot\underbrace{\frac{\cos\theta_t}{\cos\theta_i}}_{\equiv\alpha}\tilde{E}_{ot} \Rightarrow \tilde{E}_{oi} - \tilde{E}_{or} = \alpha\beta\,\tilde{E}_{ot} \quad (5)$$

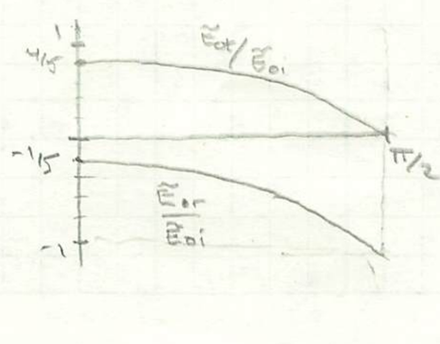
& (3) is the same as (5).

Add (4) & (5): $2\tilde{E}_{oi} = (1+\alpha\beta)\tilde{E}_{ot} \Rightarrow \tilde{E}_{ot} = \frac{2}{1+\alpha\beta}\tilde{E}_{oi}$

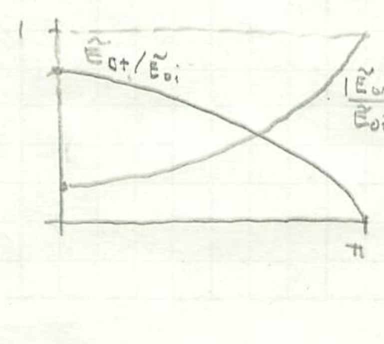
subtract " " : $2\tilde{E}_{or} = (1-\alpha\beta)\tilde{E}_{ot} \Rightarrow \tilde{E}_{or} = \frac{(1-\alpha\beta)}{1+\alpha\beta}\tilde{E}_{oi}$

w/ $\alpha = \frac{\sqrt{1-(\frac{n_1}{n_2}\sin\theta_i)^2}}{\cos\theta_i}$, at $\theta_i = 0$, $\alpha = 1$, at $\theta_i \to \pi/2$, $\alpha \to +\infty$

for $\beta = \frac{n_2}{n_1} = 3/2$,



or



For $\beta \approx \frac{n_2}{n_1}$ (meaning, here, that $\mu_1 \approx \mu_2$) we would need: $1-\alpha\beta=0$ to have a "Brewster's angle" – that means:

$$1 = \alpha\beta = \frac{n_2}{n_1}\frac{\sqrt{1-\left(\frac{n_1}{n_2}\sin\Theta_B\right)^2}}{\cos\Theta_B} \Rightarrow \left(\frac{n_1}{n_2}\right)^2(1-\sin^2\Theta_B) = 1-\left(\frac{n_1}{n_2}\right)^2\sin^2\Theta_B$$

or $\left(\frac{n_1}{n_2}\right)^2=1$, so only if $n_2=n_1$ (no change in medium).

For $\Theta_i=0$, $\alpha=1$, so $\tilde{E}_{0r} = \frac{1-\beta}{1+\beta}\tilde{E}_{0i}$ ✓ $\tilde{E}_{0t} = \frac{2}{1+\beta}\tilde{E}_{0i}$ ✓ (matches normal incidence (9.82))

The reflection coefficient is: $R = \frac{(\tilde{E}_{0r})^2}{(\tilde{E}_{0i})^2} = \frac{(1-\alpha\beta)^2}{(1+\alpha\beta)^2}$

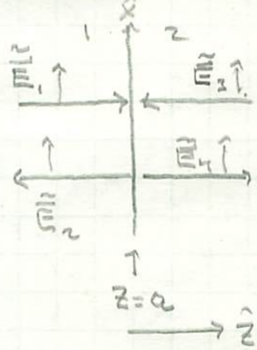
" transmission " "

$$T = \underbrace{\frac{\epsilon_2 v_2}{\epsilon_1 v_1}}_{=\beta}\left(\frac{\tilde{E}_{0t}}{\tilde{E}_{0i}}\right)^2\underbrace{\frac{\cos\Theta_t}{\cos\Theta_i}}_{=\alpha} = \alpha\beta\left(\frac{4}{(1+\alpha\beta)^2}\right)$$

checking that these add to 1:

$$R+T = \frac{(1-\alpha\beta)^2+4\alpha\beta}{(1+\alpha\beta)^2} = \frac{1-2\alpha\beta+\alpha^2\beta^2+4\alpha\beta}{(1+\alpha\beta)^2} = \frac{(1+\alpha\beta)^2}{(1+\alpha\beta)^2} = 1 \checkmark$$

Problem 2



in 1 $\begin{cases} \tilde{E}_1 = \tilde{E}_{10}\, e^{ik_1 z}\,\hat{x} & \tilde{B}_1 = \frac{\tilde{E}_{10}}{v_1}\, e^{ik_1 z}\,\hat{y} \\ \tilde{E}_2 = \tilde{E}_{20}\, e^{-ik_1 z}\,\hat{x} & \tilde{B}_2 = -\frac{\tilde{E}_{20}}{v_1}\, e^{-ik_1 z}\,\hat{y} \end{cases}$ $\qquad k_1 = \omega v_1$

in 2 $\begin{cases} \tilde{E}_3 = \tilde{E}_{30}\, e^{-ik_2 z}\,\hat{x} & \tilde{B}_3 = -\frac{\tilde{E}_{30}}{v_2}\, e^{-ik_2 z}\,\hat{y} \\ \tilde{E}_4 = \tilde{E}_{40}\, e^{+ik_2 z}\,\hat{x} & \tilde{B}_4 = \frac{\tilde{E}_{40}}{v_2}\, e^{ik_2 z}\,\hat{y} \end{cases}$ $\qquad k_2 = \omega v_2$

The electric fields are $\parallel$ to the boundary, & so suffer no discontinuity

$$\tilde{E}_{10}\, e^{ik_1 a} + \tilde{E}_{20}\, e^{-ik_1 a} = \tilde{E}_{30}\, e^{-ik_2 a} + \tilde{E}_{40}\, e^{ik_2 a} \qquad (1)$$

The magnetic fields are also $\parallel$ to the boundary, & so they have

$$\frac{1}{\mu_1}\left[\frac{\tilde{E}_{10}}{v_1} e^{ik_1 a} - \frac{\tilde{E}_{20}}{v_1} e^{-ik_1 a}\right] = \frac{1}{\mu_2}\left[-\frac{\tilde{E}_{30}}{v_2} e^{-ik_2 a} + \frac{\tilde{E}_{40}}{v_2} e^{ik_2 a}\right] \qquad (2)$$

let $\beta = \frac{\mu_1 v_1}{\mu_2 v_2}$, we can rewrite (1) & (2) as:

$$\tilde{E}_{10}\, e^{ik_1 a} - \tilde{E}_{30}\, e^{-ik_2 a} = -\tilde{E}_{20}\, e^{-ik_1 a} + \tilde{E}_{40}\, e^{ik_2 a}$$

$$\tilde{E}_{10}\, e^{ik_1 a} + \beta \tilde{E}_{30}\, e^{-ik_2 a} = \tilde{E}_{20}\, e^{-ik_1 a} + \beta \tilde{E}_{40}\, e^{ik_2 a}$$

we can write these eqns as:

$$\underbrace{\begin{pmatrix} e^{ik_1 a} & -e^{-ik_2 a} \\ e^{ik_1 a} & \beta e^{-ik_2 a} \end{pmatrix}}_{\mathbb{A}} \begin{pmatrix} \tilde{E}_{10} \\ \tilde{E}_{30} \end{pmatrix} = \underbrace{\begin{pmatrix} -e^{-ik_1 a} & e^{ik_2 a} \\ e^{-ik_1 a} & \beta e^{ik_2 a} \end{pmatrix}}_{\mathbb{B}} \begin{pmatrix} \tilde{E}_{20} \\ \tilde{E}_{40} \end{pmatrix}$$

then $$\begin{pmatrix} \tilde{E}_{20} \\ \tilde{E}_{40} \end{pmatrix} = \mathbb{B}^{-1}\mathbb{A} \begin{pmatrix} \tilde{E}_{10} \\ \tilde{E}_{30} \end{pmatrix} = \begin{pmatrix} \frac{1-\beta}{1+\beta} e^{i2k_1 a} & \frac{2\beta}{1+\beta} e^{ia(k_1-k_2)} \\ \frac{2}{1+\beta} e^{ia(k_1-k_2)} & -\frac{1-\beta}{1+\beta} e^{i2k_2 a} \end{pmatrix} \begin{pmatrix} \tilde{E}_{10} \\ \tilde{E}_{30} \end{pmatrix}$$

For $a=0$, $\tilde{E}_{30}=0$, we get: $\begin{pmatrix} \tilde{E}_{20} \\ \tilde{E}_{40} \end{pmatrix} = \begin{pmatrix} \frac{1-\beta}{1+\beta}\tilde{E}_{10} \\ \frac{2}{1+\beta}\tilde{E}_{10} \end{pmatrix}$ matching (9.82)

Problem 3

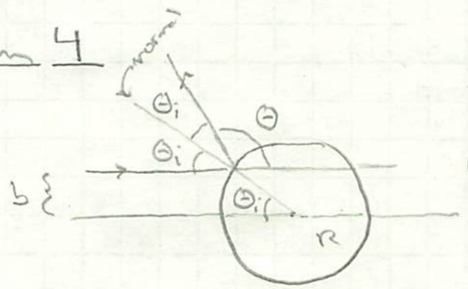
There is no force in the $\hat{x}$ direction, so the $\hat{x}$-component of momentum is conserved,
$p_{ix} = m v_1 \sin\theta_i = p_{tx} = m v_2 \sin\theta_t$, so

$$\sin\theta_t = \frac{v_1}{v_2}\sin\theta_i \Rightarrow \frac{\sin\theta_t}{\sin\theta_i} = \frac{v_1}{v_2} = \frac{n_2}{n_1}$$

we can write v_2 in terms of v_1 using conservation of energy:

$$\tfrac{1}{2} m v_1^2 = \tfrac{1}{2} m v_2^2 + U_0 \Rightarrow v_2 = \sqrt{v_1^2 - 2U_0/m}$$

Problem 4



$$\sin\theta_i = b/R \Rightarrow \theta_i = \sin^{-1}(b/R)$$

$$\Theta + 2\theta_i = \pi \Rightarrow \Theta = \pi - 2\theta_i$$

or

$$\Theta = \pi - 2\sin^{-1}(b/R)$$