Problem 1 (8.15) | | Problem Set 7

a. For the point charge, $\vec{E}_q = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}$, & the toroidal coil produces:

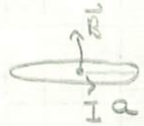
$\vec{B} = \frac{\mu_0 N I}{2\pi s}\hat{\phi}$ inside the configuration, zero outside.

Then the toroid, we have $\vec{g} = \epsilon_0 \vec{E}_q \times \vec{B} = -\frac{\mu_0 N I q}{8\pi^2 r^2 (r^2 - z^2)^{1/2}}\hat{\theta}$

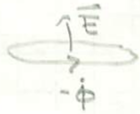
& the total momentum stored in the fields is:

$$\vec{p} = \int_{\text{toroid}} \vec{g}\, d\tau \approx +\frac{\mu_0 N I q}{8\pi^2 \cdot a^3} \cdot \underbrace{2\pi a \cdot w \cdot h}_{\text{vol of toroid}}\, \overset{\hat{z} \approx -\hat{\theta}}{\hat{z}} = +\frac{\mu_0 N I q}{4\pi}\frac{wh}{a^2}\hat{z}$$

b. The induced electric field at the center of the toroid as the current is being turned off can be found by analogy w/ the magnetic field at the center of a circular loop of wire.



$\vec{B} = \frac{\mu_0 I}{2a}\hat{z}$ & $\nabla \times \vec{B} = \mu_0 \vec{J}$ compared w/ $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
we take $\mu_0 \vec{J} \to -\dot{\vec{B}}$, or $\mu_0 I \to -\dot{\Phi}$ ← rate of flux chge.



$$\vec{E} = -\frac{\dot{\Phi}}{2a}\hat{z} \quad \& \quad \dot{\Phi} = \frac{\mu_0 N \dot{I}}{2\pi a} \cdot \overbrace{w \cdot h}^{\text{cross section area of toroid}}$$

The impulse is: $\vec{I}_m = \int_0^{t_f} \vec{F}\, dt = \int_0^{t_f} q\vec{E}\, dt = -\frac{\mu_0 N w h q}{4\pi a^2}\int_0^{t_f} \dot{I}\, dt\, \hat{z} = -\frac{\mu_0 N q w h}{4\pi a^2}\hat{z}\int_I^0 dI$

$$= \frac{\mu_0 N I q}{4\pi}\frac{wh}{a^2}\hat{z}$$

matching the momentum stored in the fields.

Problem 2 (8.9)

a.

The electric field is non-zero only between the spheres, where it is:

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}, \quad \text{then } \vec{g} = \epsilon_0 \vec{E} \times \vec{B} = \frac{Q B_0}{4\pi r^2}\overbrace{\sin\theta(-\hat{\phi})}^{= \hat{r} \times \hat{z}}$$

and

$$\vec{l} = \vec{r} \times \vec{g} = +\frac{Q B_0}{4\pi r}\sin\theta\,\hat{\theta}$$

The total angular momentum is: $\vec{L} = \int_{\text{all space}} \vec{l}\, d\tau = \int_{\text{all space}} \frac{Q B_0}{4\pi r}\sin\theta\left[\cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}\right] d\tau$

the $\hat{x}$ & $\hat{y}$ components vanish from the ϕ-integration, leaving:

$$\vec{L} = \frac{Q B_0}{4\pi} \cdot 2\pi \int_a^b \int_0^\pi \left(-\sin^3\theta\,\hat{z}\right) r\, d\theta\, dr = -\frac{Q B_0}{2}\,\frac{4}{3}\cdot\frac{1}{2}(b^2 - a^2)\hat{z}$$

$$\vec{L} = -\tfrac{1}{3} Q B_0 (b^2 - a^2)\hat{z}$$

b. As the magnetic field B is turned down from B_0 at $t=0$ to 0 at t_a, there is an induced electric field:

$$\oint_{\partial S} \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{a}$$

the induced $\vec{E} \sim \hat{\Phi}$ direction & has magnitude that could depend on s, $\vec{E} = E(s)\hat{\Phi}$. For S, take a disk of radius s:

$$\oint_{\partial S} \vec{E}\cdot d\vec{l} = E(s)\cdot 2\pi s \quad \& \quad -\frac{d}{dt}\int \vec{B}\cdot d\vec{a} = -\dot{B}(t)\cdot \pi s^2$$

so that $\vec{E} = -\frac{1}{2}\dot{B}\, s\,\hat{\Phi}$,

For the inner sphere of radius a, a point at $\vec{r}$ on its surface experiences a torque:

$$d\vec{\tau} = \vec{r}\times(dq\,\vec{E}) = a\,dq\left(-\tfrac{1}{2}\dot{B}\sqrt{a^2-z^2}\right)\hat{r}\times\hat{\Phi} = \tfrac{1}{2}a\,dq\,\dot{B}\sqrt{a^2-z^2}\cdot\hat{\Theta}, \quad dq = \frac{Q}{4\pi a^2}\cdot da$$

& integrating $d\vec{\tau}$ over the sphere, noting that the $\hat{x}$ & $\hat{y}$ components will be zero due to the Φ-integration,

$$\vec{\tau}_a = \int_0^{2\pi}\int_0^{\pi}\left(\tfrac{1}{2}a\,\dot{B}\sqrt{a^2-a^2\cos^2\Theta}\right)\frac{Q}{4\pi a^2}\left(-\sin\Theta\,\hat{z}\right)a^2\sin\Theta\, d\Theta\, d\Phi$$

$$= -\tfrac{1}{2}a^2\dot{B}\cdot\frac{Q}{4\pi a^2}\cdot a^2\cdot 2\pi\,\hat{z}\underbrace{\int_0^{\pi}\sin^3\Theta\, d\Theta}_{=4/3}\,\hat{z}$$

$$= -\tfrac{1}{3}a^2\dot{B}Q\,\hat{z}$$

And similarly for the $-Q$ sphere of radius b: $\vec{\tau}_b = \frac{1}{3}b^2\dot{B}Q\hat{z}$

From $\frac{d\vec{L}}{dt} = \vec{\tau}$, we can compute the angular momentum for both spheres:

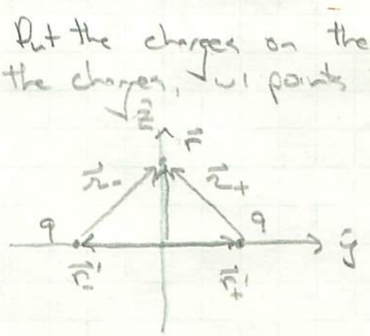
$$\vec{L}_a = \int_0^t \vec{\tau}_a\, d\bar{t} = -\tfrac{1}{3}a^2 Q\hat{z}\int_0^t \dot{B}(\bar{t})\,d\bar{t} = -\tfrac{1}{3}a^2 Q\left(B(t)-B_0\right)\hat{z}$$

& $\vec{L}_b = \frac{1}{3}b^2 Q(B(t)-B_0)\hat{z}$ so that the total angular moment at time t is:

$\vec{L} = \frac{1}{3}Q(B(t)-B_0)(b^2-a^2)\hat{z}$ — for $t \geq t_a$, $B(t)=0$, so after the $\vec{B}$-field is gone, $\vec{L} = -\frac{1}{3}QB_0(b^2-a^2)\hat{z}$, the same as the amount originally stored in the fields.

Problem 3 (8.4)

a. Put the charges on the $\hat{y}$ axis at $\vec{r}_+' = a\hat{y}$, $\vec{r}_-' = -a\hat{y}$. For the plane in between the charges, w/ points $\vec{r} = x\hat{x} + z\hat{z}$, the field due to the charge at $\vec{r}_+'$ is



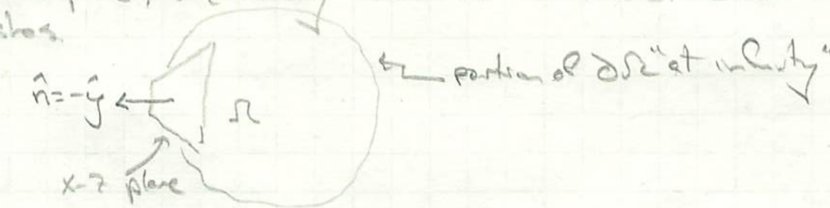
$$\vec{E}_+ = \frac{q}{4\pi\epsilon_0 \mathscr{r}_+^3}\vec{\mathscr{r}}_+ = \frac{q(x\hat{x} + z\hat{z} - a\hat{y})}{4\pi\epsilon_0 (x^2+z^2+a^2)^{3/2}}$$

$$\vec{\mathscr{r}}_+ = \vec{r} - \vec{r}_+'$$

and due to the charge at $\vec{r}_-'$, $\vec{E}_- = q\dfrac{(x\hat{x} + z\hat{z} + a\hat{y})}{4\pi\epsilon_0 (x^2+z^2+a^2)^{3/2}}$

So the total field is $\vec{E} = \vec{E}_+ + \vec{E}_- = \dfrac{q(x\hat{x} + z\hat{z})}{2\pi\epsilon_0 (x^2+z^2+a^2)^{3/2}}$,

To find the force on the charge at $\vec{r}_+'$, we'll make a domain Ω w/ boundary made up of the x-z plane, & a "bag" out at ∞ where the field (& hence stress tensor) vanishes.



At the planar surface, $d\vec{a} = -dx\,dz\,\hat{y}$, & only the T_{yy} component of the stress tensor contributes to $\overleftrightarrow{T}\cdot d\vec{a}$

$$\vec{F} = \oint_{\partial\Omega} \overleftrightarrow{T}\cdot d\vec{a} = \hat{y}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} T_{yy}\,dx\,dz \quad \text{w/ } T_{yy} \underbrace{=}_{=\epsilon_0[E_yE_y - \frac{1}{2}\delta_{yy}E^2]} -\tfrac{1}{2}\epsilon_0 E^2$$

let $x = u\cos\phi$ for $z = u\sin\phi$ "polar" coords.

$$= -\frac{q^2}{8\pi^2\epsilon_0}\hat{y}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{(x^2+z^2)}{(x^2+z^2+a^2)^3}\,dx\,dz = -\frac{q^2\hat{y}}{8\pi^2\epsilon_0}\int_0^{2\pi}\int_0^{\infty}\frac{u^2}{(u^2+a^2)^3}\,u\,du\,d\phi$$

$$= -\frac{q^2\hat{y}}{4\pi\epsilon_0}\underbrace{\int_0^{\infty}\frac{u^3}{(u^2+a^2)^3}\,du}_{=\frac{1}{4a^2}} = -\frac{q^2}{4\pi\epsilon_0(2a)^2}\hat{y}$$

✓ that is the force on the charge at $\vec{r}_+'$. ✓

b. If the charge at $\vec{r}_-'$ is $-q$, then the field at the plane is $\vec{E} = \dfrac{q(-a\hat{y})}{2\pi\epsilon_0 (x^2+z^2+a^2)^{3/2}}$

This time, $T_{yy} = \epsilon_0(E_y \cdot E_y - \frac{1}{2}\delta_{yy}E^2) = +\frac{1}{2}\epsilon_0 E^2$, & we have

$$\vec{F} = \oint_{\partial\Omega}\overleftrightarrow{T}\cdot d\vec{a} = -\frac{q^2\hat{y}\,a^2}{8\pi^2\epsilon_0}\int_0^{2\pi}\int_0^{\infty}\frac{1}{(u^2+a^2)^3}\,u\,du\,d\phi$$

$$= -\frac{q^2\hat{y}\,a^2}{4\pi\epsilon_0}\underbrace{\int_0^{\infty}\frac{u}{(u^2+a^2)^3}\,du}_{=\frac{1}{4a^4}} = -\frac{q^2}{4\pi\epsilon_0(2a)^2}\hat{y}$$ ✓