Problem 1 | | Problem Set 3

$$\int_{-\infty}^{+\infty} \delta(f(x))\,p(x)\,dx = \int_{f(-\infty)}^{f(\infty)} \frac{\delta(u)\,p(x(u))}{f'(x(u))}\,du = \int_{f(\bar{x}-\epsilon)}^{f(\bar{x}+\epsilon)} \frac{\delta(u)\,p(x(u))}{f'(x(u))}\,du$$

(let $u = f(x)$
$\Rightarrow du = f'(x)\,dx$

if $f'(\bar{x}) > 0$, then in the vicinity of $\bar{x}$, $f(\bar{x}-\epsilon) < f(\bar{x}+\epsilon)$, & $f'(\bar{x}) = |f'(\bar{x})|$

if $f'(\bar{x}) < 0$ " " " " " " $f(\bar{x}-\epsilon) > f(\bar{x}+\epsilon)$ & we
we take $f'(\bar{x}) = -|f'(\bar{x})|$ to get the limits of the integral going in the "positive" direction, so

$$\int_{-\infty}^{+\infty} \delta(f(x))\,p(x)\,dx = \int_{\min(f(\bar{x}-\epsilon),\,f(\bar{x}+\epsilon))}^{\max(f(\bar{x}-\epsilon),\,f(\bar{x}+\epsilon))} \frac{\delta(u)\,p(x(u))}{|f'(x(u))|}\,du = \frac{p(\bar{x})}{|f'(\bar{x})|} \quad (\bar{x} = x(u=0)).$$

Problem 2

The Lorentz boost has: $t = \gamma(\bar{t} + v/c^2\,\bar{x})$, $x = \gamma(\bar{x} + v\bar{t})$, $y = \bar{y}$, $z = \bar{z}$
so

$$\frac{\partial t}{\partial \bar{t}} = \gamma \quad \frac{\partial t}{\partial \bar{x}} = \gamma\, v/c^2, \quad \frac{\partial x}{\partial \bar{t}} = \gamma v \quad \frac{\partial x}{\partial \bar{x}} = \gamma, \quad \frac{\partial y}{\partial \bar{y}} = 1 \quad \frac{\partial z}{\partial \bar{z}} = 1$$

all other derivatives are zero.

$$\frac{\partial \bar{f}}{\partial \bar{t}} = \frac{\partial f}{\partial t}\frac{\partial t}{\partial \bar{t}} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{t}} + \frac{\partial f}{\partial y}\cancel{\frac{\partial y}{\partial \bar{t}}} + \frac{\partial f}{\partial z}\cancel{\frac{\partial z}{\partial \bar{t}}} = \frac{\partial f}{\partial t}\gamma + \frac{\partial f}{\partial x}\gamma v$$

& so

$$\frac{\partial^2 \bar{f}}{\partial \bar{t}^2} = \frac{\partial}{\partial \bar{t}}\left(\frac{\partial \bar{f}}{\partial \bar{t}}\right) = \frac{\partial}{\partial t}\left(\frac{\partial \bar{f}}{\partial \bar{t}}\right)\underbrace{\frac{\partial t}{\partial \bar{t}}}_{=\gamma} + \frac{\partial}{\partial x}\left(\frac{\partial \bar{f}}{\partial \bar{t}}\right)\underbrace{\frac{\partial x}{\partial \bar{t}}}_{\gamma v} + \frac{\partial}{\partial y}\left(\frac{\partial \bar{f}}{\partial \bar{t}}\right)\cancel{\frac{\partial y}{\partial \bar{t}}} + \frac{\partial}{\partial z}\left(\frac{\partial \bar{f}}{\partial \bar{t}}\right)\cancel{\frac{\partial z}{\partial \bar{t}}}$$

$$= \frac{\partial^2 f}{\partial t^2}\gamma^2 + \frac{\partial^2 f}{\partial t\,\partial x}\gamma^2 v + \frac{\partial^2 f}{\partial x\,\partial t}\gamma^2 v + \frac{\partial^2 f}{\partial x^2}\gamma^2 v^2$$

$$\frac{\partial \bar{f}}{\partial \bar{x}} = \frac{\partial f}{\partial t}\frac{\partial t}{\partial \bar{x}} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{x}} + \frac{\partial f}{\partial y}\cancel{\frac{\partial y}{\partial \bar{x}}} + \frac{\partial f}{\partial z}\cancel{\frac{\partial z}{\partial \bar{x}}} = \frac{\partial f}{\partial t}\gamma\, v/c^2 + \frac{\partial f}{\partial x}\gamma$$

$$\frac{\partial^2 \bar{f}}{\partial \bar{x}^2} = \frac{\partial}{\partial \bar{x}}\left(\frac{\partial \bar{f}}{\partial \bar{x}}\right) = \frac{\partial}{\partial t}\left(\frac{\partial \bar{f}}{\partial \bar{x}}\right)\frac{\partial t}{\partial \bar{x}} + \frac{\partial}{\partial x}\left(\frac{\partial \bar{f}}{\partial \bar{x}}\right)\frac{\partial x}{\partial \bar{x}} + \frac{\partial}{\partial y}\left(\frac{\partial \bar{f}}{\partial \bar{x}}\right)\cancel{\frac{\partial y}{\partial \bar{x}}} + \frac{\partial}{\partial z}\left(\frac{\partial \bar{f}}{\partial \bar{x}}\right)\cancel{\frac{\partial z}{\partial \bar{x}}}$$

$$= \frac{\partial^2 f}{\partial t^2}\gamma^2 v^2/c^4 + \frac{\partial^2 f}{\partial t\,\partial x}\gamma^2 v/c^2 + \frac{\partial^2 f}{\partial x\,\partial t}\gamma^2 v/c^2 + \frac{\partial^2 f}{\partial x^2}\gamma^2$$

& $\dfrac{\partial^2 \bar{f}}{\partial \bar{y}^2} = \dfrac{\partial^2 f}{\partial y^2}$ $\quad \dfrac{\partial^2 \bar{f}}{\partial \bar{z}^2} = \dfrac{\partial^2 f}{\partial z^2}$

$$-\frac{1}{c^2}\frac{\partial^2 \bar{f}}{\partial \bar{t}^2} + \frac{\partial^2 \bar{f}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{f}}{\partial \bar{y}^2} + \frac{\partial^2 \bar{f}}{\partial \bar{z}^2} = \frac{\partial^2 f}{\partial t^2}\gamma^2\left(-\frac{1}{c^2} + \frac{v^2}{c^4}\right) + \frac{\partial^2 f}{\partial x^2}\gamma^2\left(-\frac{v^2}{c^2} + 1\right) + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= -\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Problem 3

We want to evaluate $\bar{\partial}^\mu \bar{f}(\bar{t},\bar{x},\bar{y},\bar{z}) = \bar{\partial}^\mu f(t(\bar{t},\bar{x},\bar{y},\bar{z}), x(\bar{t},\bar{x},\bar{y},\bar{z}), y(\bar{t},\bar{x},\bar{y},\bar{z}), z(\bar{t},\bar{x},\bar{y},\bar{z}))$

where $ct = \gamma(c\bar{t} + v/c\,\bar{x})$, $x = \gamma(\bar{x} + v\bar{t})$, $y = \bar{y}$, $z = \bar{z}$
so that there are non-zero derivatives:

$$\frac{\partial t}{\partial \bar{t}} = \gamma \quad \frac{\partial t}{\partial \bar{x}} = \gamma\frac{v}{c^2} \quad \frac{\partial x}{\partial \bar{t}} = \gamma v \quad \frac{\partial x}{\partial \bar{x}} = \gamma \quad \frac{\partial y}{\partial \bar{y}} = 1 \quad \frac{\partial z}{\partial \bar{z}} = 1$$

For $\mu = 0$, we have

$$\bar{\partial}^0 \bar{f} = \frac{-1}{c}\frac{\partial f}{\partial \bar{t}} = \frac{-1}{c}\left[\frac{\partial f}{\partial t}\frac{\partial t}{\partial \bar{t}} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{t}} + \cancel{\frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{t}}} + \cancel{\frac{\partial f}{\partial z}\frac{\partial z}{\partial \bar{t}}}\right]$$

$$= \frac{-1}{c}\left[\gamma\frac{\partial f}{\partial t} + \gamma v\frac{\partial f}{\partial x}\right] = -\gamma\left[\frac{1}{c}\frac{\partial f}{\partial t} + \frac{v}{c}\frac{\partial f}{\partial x}\right]$$

$$\bar{\partial}^x \bar{f} = \frac{\partial f}{\partial \bar{x}} = \frac{\partial f}{\partial t}\frac{\partial t}{\partial \bar{x}} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{x}} + \cancel{\frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{x}}} + \cancel{\frac{\partial f}{\partial z}\frac{\partial z}{\partial \bar{x}}}$$

$$= \gamma\frac{v}{c}\frac{1}{c}\frac{\partial f}{\partial t} + \gamma\frac{\partial f}{\partial x} = \gamma\left[\frac{\partial f}{\partial x} + \frac{v}{c}\cdot\frac{1}{c}\frac{\partial f}{\partial t}\right]$$

$$\bar{\partial}^y \bar{f} = \frac{\partial f}{\partial \bar{y}} = \frac{\partial f}{\partial y} \qquad \bar{\partial}^z \bar{f} = \frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial z}$$

We can write these relations as:

$$\bar{\partial}^0 \bar{f} = \gamma[\partial^0 f - v/c\,\partial^x f]$$

$$\bar{\partial}^x \bar{f} = \gamma[\partial^x f - v/c\,\partial^0 f]$$

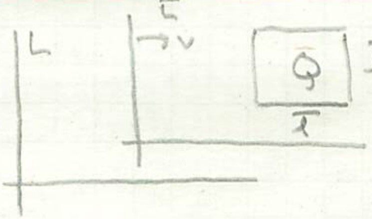
$$\bar{\partial}^y \bar{f} = \partial^y f$$

$$\bar{\partial}^z \bar{f} = \partial^z f$$

$$\Rightarrow \begin{pmatrix} \bar{\partial}^0 \bar{f} \\ \bar{\partial}^x \bar{f} \\ \bar{\partial}^y \bar{f} \\ \bar{\partial}^z \bar{f} \end{pmatrix} = \begin{pmatrix} \gamma & -v/c\,\gamma & 0 & 0 \\ -v/c\,\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \partial^0 f \\ \partial^x f \\ \partial^y f \\ \partial^z f \end{pmatrix}$$

or $\bar{\partial}^\mu \bar{f} = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu (\partial^\nu f)$

Problem 4




in $\bar{L}$: $\bar{\rho} = \frac{Q}{\bar{\ell}^3}$, $\bar{J} = 0$,

In L: the box has vol. $\bar{\ell}^2 \cdot \frac{1}{\gamma}\bar{\ell}$

y + z sides are the same

length contraction in x direction

so $\rho = \frac{\gamma Q}{\bar{\ell}^3} = \gamma\bar{\rho}$, $J = \rho v = \gamma \bar{\rho} v$

If $J^\mu \doteq \begin{pmatrix} \rho c \\ \vec{J} \end{pmatrix}$ was a 4-vector, we'd have: $\rho c = \gamma(\bar{\rho} c + v/c \cancel{\bar{J}})$

$\Downarrow$

$\rho = \gamma \bar{\rho}$ ✓

and $J = \gamma(\cancel{\bar{J}} + v/c\, \bar{\rho} c) = \gamma \bar{\rho} v$ ✓ + $J^\mu$ is a 4-vector.

In $\bar{L}$: $-\bar{\rho}^2 c^2 + \bar{J}^2 = -\bar{\rho}^2 c^2 = -(Q/\bar{\ell}^3)^2 c^2$

In L: $-\rho^2 c^2 + J^2 = -\gamma^2 \bar{\rho}^2 c^2 + \gamma^2 \bar{\rho}^2 v^2 = -c^2 \gamma^2 \bar{\rho}^2 (1 - v^2/c^2)$

$= -\bar{\rho}^2 c^2 = -(Q/\bar{\ell}^3)^2 c^2$,

so we have $-\bar{\rho}^2 c^2 + \bar{J}^2 = -\rho^2 c^2 + J^2$ ✓

Problem 5

In the rest frame of the line of charge: $\bar{V} = -\frac{\bar{\lambda}}{2\pi\epsilon_0}\log(\bar{s}/\bar{s}_0)$, $\bar{A} = 0$

For the moving line of charge: $V = -\frac{\gamma\bar{\lambda}}{2\pi\epsilon_0}\log(s/s_0)$, $\vec{A} = -\frac{\gamma\bar{\lambda} v}{2\pi \mu_0^{-1}}\log(s/s_0)\hat{\phi}$

+ $\bar{s} = s$, $\bar{s}_0 = s_0$, those directions ($r$ $\hat{\phi}$) are $\perp$ to the boost, so

$V = \gamma \bar{V}$ $\qquad A = \gamma \bar{V} v \epsilon_0 \mu_0 = \gamma \bar{V} v/c^2$

If $A^\mu \doteq \begin{pmatrix} V/c \\ A \end{pmatrix}$ is a 4-vec., we'd have: $V = \gamma(\bar{V} + v/c\cancel{\bar{A}}) = \gamma \bar{V}$ ✓

$A = \gamma(\cancel{\bar{A}} + v/c^2\, \bar{V}) = \gamma \bar{V} v/c^2$ ✓

In $\bar{L}$: $-\frac{\bar{V}^2}{c^2} + \bar{A}^2 = -\frac{\bar{V}^2}{c^2}$ $\qquad$ In L: $-\frac{V^2}{c^2} + A^2 = -\frac{\gamma^2 \bar{V}^2}{c^2} + \gamma^2 \bar{V}^2 v^2/c^4$

$= -\frac{\bar{V}^2}{c^2}\gamma^2(1 - v^2/c^2) = -\frac{\bar{V}^2}{c^2}$

so $-\frac{\bar{V}^2}{c^2} + \bar{A}^2 = -\frac{V^2}{c^2} + A^2$ ✓