Problem 1 | | Problem Set 12

$$\partial_\mu(\mathcal{L} M^\mu{}_\nu x^\nu) = (\partial_\mu \mathcal{L}) M^\mu{}_\nu x^\nu + \mathcal{L}\,\partial_\mu(M^\mu{}_\nu x^\nu)$$

& $M^\mu{}_\nu$ does not depend on position, so

$$\partial_\mu(M^\mu{}_\nu x^\nu) = M^\mu{}_\nu \underbrace{\frac{\partial x^\nu}{\partial x^\mu}}_{=\delta^\nu_\mu} = M^\mu{}_\mu = M^0{}_0 + M^1{}_1 + M^2{}_2 + M^3{}_3$$

but both boosts & rotations have infinitesimal matrices that have zeros along the diagonal, so $M^\mu{}_\mu = 0$, &

$$\partial_\mu(\mathcal{L} M^\mu{}_\nu x^\nu) = (\partial_\mu \mathcal{L}) M^\mu{}_\nu x^\nu$$

Problem 2

The expressions in i & iv are valid.

ii $A^\mu B_\mu C^\mu = K^\nu$ has 3 indices on the left, leading to ambiguity in the summation. On the right, there is an unmatched open ν index.

Fix: $A^\nu B_\mu C^\mu = K^\nu$

iii $W^\mu = V^\nu$ has an open μ on the left, an open ν on the right. both are unmatched.

fix: $W^\mu = V^\mu$

v. $A^\mu B^\mu = S$ has both indices up.

fix $A^\mu B_\mu = S$.

Problem 3

In $\bar{L}$, $\vec{\bar{E}} = \dfrac{\bar{\lambda}}{2\pi\epsilon_0 \bar{s}}\hat{s}$ & $\vec{\bar{B}} = 0$ so $2(-\bar{E}^2/c^2 + \bar{B}^2) = -\dfrac{\bar{\lambda}^2}{2\pi^2\epsilon_0^2\bar{s}^2c^2}$

& $-4\vec{\bar{E}}\cdot\vec{\bar{B}}/c = 0$.

In L: $\vec{E} = \dfrac{\lambda}{2\pi\epsilon_0 s}\hat{s}$ & $\vec{B} = \dfrac{\mu_0\lambda v}{2\pi s}\hat{\phi}$ w/ $\lambda = \gamma\bar{\lambda}$, $s = \bar{s}$, so

$$= \frac{\bar{\lambda}\gamma}{2\pi\epsilon_0\bar{s}}\hat{s} \qquad = \frac{\mu_0\bar{\lambda}\gamma v}{2\pi\bar{s}}\hat{\phi}$$

and here, ($\mu_0 = \frac{1}{\epsilon_0 c^2}$)

$$2(-E^2/c^2 + B^2) = -\frac{\bar{\lambda}^2\gamma^2}{2\pi^2\epsilon_0^2\bar{s}^2c^2} + \frac{\mu_0^2\bar{\lambda}^2\gamma^2 v^2}{2\pi^2\bar{s}^2} = -\frac{\bar{\lambda}^2}{2\pi^2\epsilon_0^2\bar{s}^2c^2}\overbrace{\left[1 - \frac{v^2}{c^2}\right]\gamma^2}^{=1}$$

$$= \frac{-\bar{\lambda}^2}{2\pi^2\epsilon_0^2\bar{s}^2c^2}$$

& $-4\vec{E}\cdot\vec{B}/c = 0$, matching the result in $\bar{L}$.

match

Problem 4

For $\tilde{\vec{E}}(f) = \int_{-\infty}^{+\infty} \vec{E}(t) e^{i2\pi f t} dt$, we have

$$|\tilde{\vec{E}}|^2 = \tilde{\vec{E}}(f) \cdot \tilde{\vec{E}}^*(f) = \left[\int_{-\infty}^{+\infty} \vec{E}(t) e^{i2\pi f t} dt\right] \cdot \left[\int_{-\infty}^{+\infty} \vec{E}(\bar{t}) e^{-i2\pi f \bar{t}} d\bar{t}\right]$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \vec{E}(t) \cdot \vec{E}(\bar{t}) e^{i2\pi f(t-\bar{t})} dt\, d\bar{t}$$

so

$$\int_{-\infty}^{+\infty} \tilde{\vec{E}}(f)^2 df = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \vec{E}(t) \cdot \vec{E}(\bar{t}) \underbrace{\left[\int_{-\infty}^{+\infty} e^{i2\pi f(t-\bar{t})} df\right]}_{=\delta(t-\bar{t})} d\bar{t}\, dt$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \vec{E}(t) \cdot \vec{E}(\bar{t}) \delta(t-\bar{t}) d\bar{t}\, dt$$

$$= \int_{-\infty}^{+\infty} \vec{E}(t)^2 dt \quad \checkmark$$

Problem 5

In $\bar{L}$, $\bar{F}^{\mu\nu} \doteq \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{B} & 0 \\ 0 & -\bar{B} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

& $F^{\mu\nu} = \tilde{\Lambda}^{\mu}{}_{\alpha} \tilde{\Lambda}^{\nu}{}_{\beta} \bar{F}^{\alpha\beta}$ w/ $\tilde{\Lambda}^{\rho}{}_{\sigma} \doteq \begin{pmatrix} \gamma & v/c\,\gamma & 0 & 0 \\ v/c\,\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ (the "inverse" Lorentz boost in the x-direction)

$F^{01} = \tilde{\Lambda}^0{}_\alpha \tilde{\Lambda}^1{}_\beta \bar{F}^{\alpha\beta} = (\tilde{\Lambda}^0{}_1 \tilde{\Lambda}^1{}_2 - \tilde{\Lambda}^0{}_2 \tilde{\Lambda}^1{}_1)\bar{B} = 0$

$F^{02} = \tilde{\Lambda}^0{}_\alpha \tilde{\Lambda}^2{}_\beta \bar{F}^{\alpha\beta} = (\tilde{\Lambda}^0{}_1 \tilde{\Lambda}^2{}_2 - \tilde{\Lambda}^0{}_2 \tilde{\Lambda}^2{}_1)\bar{B} = v/c\,\gamma \bar{B}$

$F^{03} = \tilde{\Lambda}^0{}_\alpha \tilde{\Lambda}^3{}_\beta \bar{F}^{\alpha\beta} = (\tilde{\Lambda}^0{}_1 \tilde{\Lambda}^3{}_2 - \tilde{\Lambda}^0{}_2 \tilde{\Lambda}^3{}_1)\bar{B} = 0$

$F^{12} = \tilde{\Lambda}^1{}_\alpha \tilde{\Lambda}^2{}_\beta \bar{F}^{\alpha\beta} = (\tilde{\Lambda}^1{}_1 \tilde{\Lambda}^2{}_2 - \tilde{\Lambda}^1{}_2 \tilde{\Lambda}^2{}_1)\bar{B} = \gamma \bar{B}$

$F^{13} = \tilde{\Lambda}^1{}_\alpha \tilde{\Lambda}^3{}_\beta \bar{F}^{\alpha\beta} = (\tilde{\Lambda}^1{}_1 \tilde{\Lambda}^3{}_2 - \tilde{\Lambda}^1{}_2 \tilde{\Lambda}^3{}_1)\bar{B} = 0$

$F^{23} = \tilde{\Lambda}^1{}_\alpha \tilde{\Lambda}^3{}_\beta \bar{F}^{\alpha\beta} = (\tilde{\Lambda}^1{}_1 \tilde{\Lambda}^3{}_2 - \tilde{\Lambda}^1{}_2 \tilde{\Lambda}^3{}_1)\bar{B} = 0$

So in L, $\vec{B} = \gamma \bar{B} \hat{z}$ & $\vec{E} = v\gamma\bar{B}\hat{y}$. The force on q, moving through L w/ $\vec{v} = v\hat{x}$, is

$$\vec{F} = q\vec{E} + q\vec{v}\times\vec{B} = qv\gamma\bar{B}\hat{y} - qv\gamma\bar{B}\hat{y} = 0$$

no net force.