Problem 1

Problem Set 11

a. $A^{\mu\nu} = F^{\mu\alpha} b_\alpha G^{\nu\beta} f_\beta$

b. $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \sum_{\alpha,\beta=0}^{3} g_{\alpha\beta} R^{\alpha\beta} = \kappa T_{\mu\nu}$ for $\mu,\nu = 0,1,2,3$

c. $A^\mu \doteq \begin{pmatrix} A^0 \\ \vec{A} \end{pmatrix}$ $B_\mu \doteq \begin{pmatrix} -B^0 \\ \vec{B} \end{pmatrix}$ so $A^\mu B_\mu = -A^0 B^0 + \vec{A}\cdot\vec{B}$

same $\Rightarrow A^\mu B_\mu = A_\mu B^\mu$

$A_\mu \doteq \begin{pmatrix} -A^0 \\ \vec{A} \end{pmatrix}$, $B^\mu \doteq \begin{pmatrix} B^0 \\ \vec{B} \end{pmatrix}$ so $A_\mu B^\mu = -A^0 B^0 + \vec{A}\cdot\vec{B}$

Problem 2

We saw: $F^{\mu\nu} \doteq \begin{pmatrix} 0 & E^x/c & E^y/c & E^z/c \\ -E^x/c & 0 & B^z & -B^y \\ -E^y/c & -B^z & 0 & B^x \\ -E^z/c & B^y & -B^x & 0 \end{pmatrix}$, & we know $\eta_{\mu\nu} \doteq \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

then: $F_{\alpha\beta} \equiv \eta_{\alpha\mu}\eta_{\beta\nu} F^{\mu\nu}$ has $F_{\beta\alpha} = \eta_{\alpha\nu}\eta_{\beta\mu} F^{\mu\nu} = -\eta_{\alpha\nu}\eta_{\beta\mu} F^{\nu\mu} = -F_{\alpha\beta}$

so it is also antisymmetric - then we know all diagonal entries are zero, & we only need to compute the upper-Δ'r piece:

$F_{01} = \eta_{0\mu}\eta_{1\nu} F^{\mu\nu} = -F^{01}$, $F_{02} = \eta_{0\mu}\eta_{2\nu} F^{\mu\nu} = -F^{02}$, $F^{03} = \eta_{0\mu}\eta_{3\nu} F^{\mu\nu} = -F^{03}$

$F_{12} = \eta_{1\mu}\eta_{2\nu} F^{\mu\nu} = F^{12}$, $F_{13} = \eta_{1\mu}\eta_{3\nu} F^{\mu\nu} = F^{13}$, $F_{23} = \eta_{2\mu}\eta_{3\nu} F^{\mu\nu} = F^{23}$

so $F_{\mu\nu} \doteq \begin{pmatrix} 0 & -E^x/c & -E^y/c & -E^z/c \\ E^x/c & 0 & B^z & -B^y \\ E^y/c & -B^z & 0 & B^x \\ E^z/c & B^y & -B^x & 0 \end{pmatrix}$

$$\begin{aligned} F^{\mu\nu}F_{\mu\nu} &= F^{01}F_{01} + F^{02}F_{02} + F^{03}F_{03} + F^{10}F_{10} + F^{20}F_{20} + F^{30}F_{30} \\ &+ F^{12}F_{12} + F^{13}F_{13} + F^{23}F_{23} + F^{21}F_{21} + F^{31}F_{31} + F^{32}F_{32} \\ &= -E^2/c^2 - E^2/c^2 + B^2 + B^2 = 2(-E^2/c^2 + B^2) \end{aligned}$$

Problem 3

$G^{\mu\nu} = 1/2\, \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$ & $G^{\mu\nu} = -G^{\nu\mu}$ so we know the dgl. components are zero, & we only need the upper Δ'r piece

$G^{01} = 1/2\, \epsilon^{01\alpha\beta} F_{\alpha\beta} = 1/2\, \epsilon^{0123} F_{23} + 1/2\, \epsilon^{0132} F_{32} = F_{23} = B^x$

$G^{02} = 1/2\, \epsilon^{02\alpha\beta} F_{\alpha\beta} = 1/2\, \epsilon^{0213} F_{13} + 1/2\, \epsilon^{0231} F_{31} = F_{31} = B^y$

$G^{03} = 1/2\, \epsilon^{03\alpha\beta} F_{\alpha\beta} = 1/2\, \epsilon^{0312} F_{12} + 1/2\, \epsilon^{0321} F_{21} = F_{12} = B^z$

Problem 3 (continued)

$$G^{12} = \tfrac{1}{2}\epsilon^{12\alpha\beta}F_{\alpha\beta} = \tfrac{1}{2}\epsilon^{1203}F_{03} + \tfrac{1}{2}\epsilon^{1230}F_{30} = F_{03} = -E^z/c$$

$$G^{13} = \tfrac{1}{2}\epsilon^{13\alpha\beta}F_{\alpha\beta} = \tfrac{1}{2}\epsilon^{1302}F_{02} + \tfrac{1}{2}\epsilon^{1320}F_{20} = F_{20} = E^y/c$$

$$G^{23} = \tfrac{1}{2}\epsilon^{23\alpha\beta}F_{\alpha\beta} = \tfrac{1}{2}\epsilon^{2301}F_{01} + \tfrac{1}{2}\epsilon^{2310}F_{10} = F_{01} = -E^x/c$$

So $G^{\mu\nu} \doteq \begin{pmatrix} 0 & B^x & B^y & B^z \\ -B^x & 0 & -E^z/c & E^y/c \\ -B^y & E^z/c & 0 & -E^x/c \\ -B^z & -E^y/c & E^x/c & 0 \end{pmatrix}$

From the relation between $F^{\mu\nu}$ & $F_{\mu\nu}$, we can write down the elts of $G_{\mu\nu}$:

$$G_{\mu\nu} \doteq \begin{pmatrix} 0 & -B^x & -B^y & -B^z \\ B^x & 0 & -E^z/c & E^y/c \\ B^y & E^z/c & 0 & -E^x/c \\ B^z & -E^y/c & E^x/c & 0 \end{pmatrix}$$

$$G^{\mu\nu}G_{\mu\nu} = 2(-B^2 + E^2/c^2)$$

&

$$F^{\mu\nu}G_{\mu\nu} = 2(-\tfrac{1}{c}\vec{E}\cdot\vec{B}) + 2(-\tfrac{1}{c}\vec{E}\cdot\vec{B}) = -\tfrac{4}{c}\vec{E}\cdot\vec{B}$$

Problem 4

$$T_{\alpha\beta} = \underbrace{\tfrac{1}{2}(T_{\alpha\beta} + T_{\beta\alpha})}_{\text{sym.}} + \underbrace{\tfrac{1}{2}(T_{\alpha\beta} - T_{\beta\alpha})}_{\text{antisym}} = S_{\alpha\beta} + A_{\alpha\beta}$$

For $A_{\mu\nu}$ w/ $A_{\nu\mu} = -A_{\mu\nu}$ & $S^{\alpha\beta}$ w/ $S^{\beta\alpha} = S^{\alpha\beta}$, we have

$$A_{\rho\sigma}S^{\rho\sigma} \underset{\substack{\uparrow \\ \text{using sym.} \\ \text{props}}}{=} -A_{\sigma\rho}S^{\sigma\rho} \underset{\substack{\uparrow \\ \text{relabel summation} \\ \text{indices: } \sigma\to\alpha,\ \rho\to\beta}}{=} -A_{\alpha\beta}S^{\alpha\beta} \underset{\substack{\uparrow \\ \text{relabel} \\ \text{summation} \\ \text{indices: } \alpha\to\rho,\ \beta\to\sigma}}{=} -A_{\rho\sigma}S^{\rho\sigma}$$

& we have $A_{\rho\sigma}S^{\rho\sigma} = -A_{\rho\sigma}S^{\rho\sigma} \Rightarrow A_{\rho\sigma}S^{\rho\sigma} = 0$.

Then: $T^{\mu\nu}A_{\mu\nu} = (T_s^{\mu\nu} + T_a^{\mu\nu})A_{\mu\nu} = \cancel{T_s^{\mu\nu}A_{\mu\nu}} + T_a^{\mu\nu}A_{\mu\nu}$ ✓

Problem 5

For $T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - \frac{1}{2}\eta^{\mu\nu}(\partial_\alpha \phi \partial^\alpha \phi)$ w/ $\partial_\rho \partial^\rho \phi = 0$
we have:

$$\partial_\mu T^{\mu\nu} = (\underbrace{\partial_\mu \partial^\mu \phi}_{\text{by field eqn.}})\partial^\nu \phi + \partial^\mu \phi (\partial_\mu \partial^\nu \phi) - \eta^{\mu\nu}(\partial_\mu \partial^\alpha \phi)\partial_\alpha \phi$$

$$= \partial^\alpha \phi (\partial_\alpha \partial^\nu \phi) - \partial^\alpha \phi (\partial^\nu \partial_\alpha \phi) = 0 \checkmark$$

Problem 6

For $A^\mu = P^\mu e^{ik_\alpha x^\alpha}$, we have $\partial_\sigma A^\mu = P^\mu e^{ik_\alpha x^\alpha} \frac{\partial}{\partial x^\sigma}(ik_\alpha x^\alpha)$

$$= P^\mu e^{ik_\alpha x^\alpha} ik_\alpha \underbrace{\frac{\partial x^\alpha}{\partial x^\sigma}}_{\delta^\alpha_\sigma} = ik_\sigma P^\mu e^{ik_\alpha x^\alpha}$$

so $\partial_\gamma A^\gamma = 0 \Rightarrow ik_\gamma P^\gamma e^{ik_\alpha x^\alpha} = 0 \Rightarrow k_\gamma P^\gamma = 0$

We also have: $\partial^\sigma \partial_\sigma A^\mu = \partial^\sigma (ik_\sigma P^\mu e^{ik_\alpha x^\alpha}) = -k^\sigma k_\sigma P^\mu e^{ik_\alpha x^\alpha} = 0$
w/

$$k^\sigma k_\sigma = 0$$