Problem 1

a. $\vec{E} = -\frac{\partial \vec{A}}{\partial t} = -\omega P_0 \sin(kx-\omega t)\hat{y}$; $\vec{B} = \frac{\partial A_y}{\partial x}\hat{z} = -kP_0\sin(kx-\omega t)\hat{z}$ w/ $\omega = kc$

$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0}k\omega P_0^2\sin^2(kx-\omega t)\hat{x} = \frac{1}{\mu_0 c}\omega^2 P_0^2 \sin^2(kx-\omega t)\hat{x}$

the intensity is the time average of $\vec{S}$: $\vec{I} = \langle \vec{S} \rangle = \frac{1}{2}\frac{\omega^2}{\mu_0 c}P_0^2\hat{x}$

b. The potential, in $\bar{L}$, is: $\vec{A} = P_0\cos\left(k\gamma(\bar{x}+v\bar{t}) - \omega\gamma(\bar{t}+v/c^2\,\bar{x})\right)\hat{y}$

$\overset{!}{=} P_0\cos\left(\bar{x}(k\gamma-\omega\gamma v/c^2) - \bar{t}(\omega\gamma - k\gamma v)\right)\hat{y}$

$\overset{!}{=} P_0\cos(\bar{k}\bar{x}-\bar{\omega}\bar{t})\hat{y}$ w/ $\bar{k} = \gamma k(1-v/c)$

$\bar{\omega} = \gamma\omega(1-v/c)$

The electric field is: $\bar{\vec{E}} = -\frac{\partial \bar{\vec{A}}}{\partial \bar{t}} = -\bar{\omega}P_0\sin(\bar{k}\bar{x}-\bar{\omega}\bar{t})\hat{y}$,

$\bar{\vec{B}} = \frac{\partial \bar{A}_y}{\partial \bar{x}}\hat{z} = -\bar{k}P_0\sin(\bar{k}\bar{x}-\bar{\omega}\bar{t})\hat{z}$

w/ $\bar{\vec{S}} = \frac{1}{\mu_0}\bar{\vec{E}}\times\bar{\vec{B}} = \frac{1}{\mu_0 c}\bar{\omega}^2 P_0^2\sin^2(\bar{k}\bar{x}-\bar{\omega}\bar{t})\hat{x}$ & $\bar{\vec{I}} = \langle\bar{\vec{S}}\rangle = \frac{1}{2}\frac{\bar{\omega}^2}{\mu_0 c}P_0^2\hat{x}$

the wave speed is: $\frac{\bar{\omega}}{\bar{k}} = \frac{\omega}{k} = c$ ✓

the frequency in $\bar{L}$ is $\bar{\omega} = \gamma\omega(1-v/c)$

$\overset{!}{=} \frac{(1-v/c)\omega}{\sqrt{1-v^2/c^2}} = \sqrt{\frac{1-v/c}{1+v/c}}\,\omega = \frac{\sqrt{c-v}}{\sqrt{c+v}}\,\omega$

Doppler shift

c. as $v \to c$: $\bar{\omega} \to 0$ & $\bar{\vec{I}} = \frac{1}{2}\frac{\bar{\omega}^2}{\mu_0 c}P_0^2\hat{x} \longrightarrow 0$, & $\bar{\vec{E}} \to 0$, $\bar{\vec{B}} \to 0$

Problem 2

Start w/ $\partial^\mu A^\nu$ for $\mu = 0 \to 3$, $\nu = 0 \to 3$, we have non-zero components: (in tableaux form):

ν \ $\mu =$	0	1	2	3	
0	0	$-\frac{1}{c}\frac{\partial A^x}{\partial t}$	$-\frac{1}{c}\frac{\partial A^y}{\partial t}$	$-\frac{1}{c}\frac{\partial A^z}{\partial t}$	} looks like $-\frac{1}{c}\frac{\partial \vec{A}}{\partial t}$
1	$\frac{1}{c}\frac{\partial V}{\partial x}$	0	$\frac{\partial A^y}{\partial x}$	$\frac{\partial A^z}{\partial x}$	← pieces of the curl
2	$\frac{1}{c}\frac{\partial V}{\partial y}$	$\frac{\partial A^x}{\partial y}$	0	$\frac{\partial A^z}{\partial y}$	
3	$\frac{1}{c}\frac{\partial V}{\partial z}$	$\frac{\partial A^x}{\partial z}$	$\frac{\partial A^y}{\partial z}$	0	
	looks like $\frac{1}{c}\nabla V$				

We need to mix the top row w/ the left column to get: $-\nabla V - \frac{\partial \vec{A}}{\partial t}$, suggesting we take:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \doteq \begin{pmatrix} 0 & -\frac{1}{c}\left(\frac{\partial V}{\partial x}+\frac{\partial A^x}{\partial t}\right) & -\frac{1}{c}\left(\frac{\partial V}{\partial y}+\frac{\partial A^y}{\partial t}\right) & -\frac{1}{c}\left(\frac{\partial V}{\partial z}+\frac{\partial A^z}{\partial t}\right) \\ +\frac{1}{c}\left(\frac{\partial V}{\partial x}+\frac{\partial A^x}{\partial t}\right) & 0 & \frac{\partial A^y}{\partial x}-\frac{\partial A^x}{\partial y} & \frac{\partial A^z}{\partial x}-\frac{\partial A^x}{\partial z} \\ +\frac{1}{c}\left(\frac{\partial V}{\partial y}+\frac{\partial A^y}{\partial t}\right) & -\left(\frac{\partial A^y}{\partial x}-\frac{\partial A^x}{\partial y}\right) & 0 & \frac{\partial A^z}{\partial y}-\frac{\partial A^y}{\partial z} \\ +\frac{1}{c}\left(\frac{\partial V}{\partial z}+\frac{\partial A^z}{\partial t}\right) & -\left(\frac{\partial A^z}{\partial x}-\frac{\partial A^x}{\partial z}\right) & -\left(\frac{\partial A^z}{\partial y}-\frac{\partial A^y}{\partial z}\right) & 0 \end{pmatrix}$$

So

$$\partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & \frac{1}{c}E^x & \frac{1}{c}E^y & \frac{1}{c}E^z \\ -\frac{1}{c}E^x & 0 & B^z & -B^y \\ -\frac{1}{c}E^y & -B^z & 0 & B^x \\ -\frac{1}{c}E^z & B^y & -B^x & 0 \end{pmatrix}$$

gives 2 copies of $\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$ + $\vec{B} = \nabla \times \vec{A}$.

Problem 3

Taking $x(t) \to x(t) + \epsilon f(t)$

$$S[x+\epsilon f] = \int_0^T L(x+\epsilon f, \dot{x}+\epsilon \dot{f}, \ddot{x}+\epsilon \ddot{f})\,dt \approx \underbrace{\int_0^T L(x,\dot{x},\ddot{x})\,dt}_{S[x(t)]} + \underbrace{\epsilon \int_0^T \left[\frac{\partial L}{\partial x} f + \frac{\partial L}{\partial \dot{x}}\dot{f} + \frac{\partial L}{\partial \ddot{x}}\ddot{f}\right] dt}_{\delta S}$$

w/

$$\int_0^T \frac{\partial L}{\partial \dot{x}} \dot{f}\,dt = \cancel{\frac{\partial L}{\partial \dot{x}} f \Big|_0^T} - \int_0^T f \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\,dt = -\int_0^T f \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\,dt$$

$$\int_0^T \frac{\partial L}{\partial \ddot{x}} \ddot{f}\,dt = \cancel{\frac{\partial L}{\partial \ddot{x}} \dot{f} \Big|_0^T} - \int_0^T \dot{f} \frac{d}{dt}\frac{\partial L}{\partial \ddot{x}}\,dt = -\left[f \frac{d}{dt}\frac{\partial L}{\partial \ddot{x}} \Big|_0^T - \int_0^T f \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}}\,dt \right]$$

$$= \int_0^T f \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}}\,dt$$

using these in δS gives:

$$\delta S = \epsilon \int_0^T \left[\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}}\right] f\,dt$$

+ if we require $\delta S = 0\ \forall f$, we have:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}} = 0$$

for $L(x,\dot{x},\ddot{x},\dddot{x})$ I expect:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}} - \frac{d^3}{dt^3}\frac{\partial L}{\partial \dddot{x}} = 0$$

Problem 4

a. For $L = \frac{1}{2} m \dot{x}(t)^2 - U(x(t)) + F(x(t))\dot{x}(t)$
we have:

$$\frac{\partial L}{\partial x} = -\frac{dU}{dx} + \frac{dF}{dx}\dot{x} \qquad \frac{\partial L}{\partial \dot{x}} = m\dot{x} + F$$

w/

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}\right] = m\ddot{x} + \frac{dF}{dx}\dot{x}, \text{ so } -\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{\partial L}{\partial x} = 0 \Rightarrow -m\ddot{x} - \underbrace{\frac{dF}{dx}\dot{x} - \frac{dU}{dx} + \frac{dF}{dx}\dot{x}}_{\text{cancel}} = 0$$

so $m\ddot{x} = -\frac{dU}{dx}$, the addition of $F\dot{x}$ does not change the eqn.s of motion.

b. A term that looked like $\dddot{x}$ could only come from a Lagrangian w/

1. $\frac{\partial L}{\partial \dot{x}} \sim \ddot{x}$ (then $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \sim \dddot{x}$), 2. $\frac{\partial L}{\partial x} \sim \dddot{x}$ or (referring to problem 3)

3. $\frac{\partial L}{\partial \ddot{x}} \sim \dot{x}$ (then $\frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}} \sim \dddot{x}$)

For 1., we'd need a term like $\dot{x}\ddot{x}$, so using the eqn. of motion from problem 3:
w/

$$L = \frac{1}{2}m\dot{x}^2 - U(x) + \alpha\dot{x}\ddot{x}$$

we have:

$$\frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{\partial L}{\partial x} = 0 \Rightarrow \alpha\dddot{x} - (m\ddot{x} + \alpha\dddot{x}) - \frac{dU}{dx} = 0$$

cancel

so this type of term will not produce a $\dddot{x}$ in the eqn. of motion.

For 2. we need a term of the form: $x\dddot{x}$, but now $L = \frac{1}{2}m\dot{x}^2 - U(x) + \alpha x \dddot{x}$ has eqn. of motion:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}} - \frac{d^3}{dt^3}\frac{\partial L}{\partial \dddot{x}} = 0 \Rightarrow -\frac{dU}{dx} + \alpha\dddot{x} - m\ddot{x} - \alpha\dddot{x} = 0 \Rightarrow m\ddot{x} = -\frac{dU}{dx}, \text{ still}$$

no $\dddot{x}$ term.

Finally, 3. requires a term like $\dot{x}\ddot{x}$, as w/ 1., which we already know won't work.