Midterm Examination

Electrodynamics II
Physics 322

Due Date: March 8th, 2024

Instructions: There are four problems (ten points each). You may use any online/book resource you like (cite where appropriate) but may not work with other people on the exam, nor ask any real-time questions of the internet (i.e. no ChatGPT or similar). If you have any questions about what resources are available, or have any questions about what a problem is asking, please contact the course instructor. The examination is due by noon on Friday, March 8th (turn in via gradescope).

Problem 1

In two dimensions, working in polar coordinates s (radial) and ϕ (angular), the Laplacian of a function $f(s, \phi)$ is

$$\nabla^2 f(s, \phi) = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial f}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 f}{\partial \phi^2}. \tag{1}$$

Suppose we have discovered that the structure of E&M in two dimensions is similar to its structure in three, with an electrostatic potential V governed by

$$\nabla^2 V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon}, \tag{2}$$

where ρ is a charge-per-unit-area (area is the "volume" in two dimensions).

a. What are the units of the electric constant ϵ if the electric field $\mathbf{E} = -\nabla V$ causes a force on a charge q given by $\mathbf{F} = q\mathbf{E}$ as in three dimensions?

b. The general integral solution to (2) is

$$V(\mathbf{r}) = \int_{\text{all space}} G(\mathbf{r}, \mathbf{r}')\rho(\mathbf{r}')d\tau' \tag{3}$$

(here $d\tau'$ is the volume element in two dimensions, so has units of area). Find the Green's function for this theory, i.e. what is $G(\mathbf{r}, \mathbf{r}') =$?

c. Find the electric field for a point charge that is located at $\mathbf{r}' = x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}}$ in the two-dimensional plane. Write the electric field, $\mathbf{E}(\mathbf{r})$ in Cartesian coordinates (i.e. assume that $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$).

Problem 2

a. Which radiates more, a circular loop of wire of radius b with time-varying (neutral) current $I(t)$, or a square loop of wire with side-length $\pi b/2$ that has the same time-varying current $I(t)$? What is the ratio of the (non-relativistic) power radiated: $\frac{P_{\text{bigger}}}{P_{\text{smaller}}} =$?

b. Which radiates (time-averaged) more, a charge q moving along the $\hat{\mathbf{z}}$ axis with $\mathbf{w}(t) = d\cos(\omega t)\hat{\mathbf{z}}$, or one moving with $\mathbf{w}(t) = d\cos^3(\omega t)\hat{\mathbf{z}}$? What is the ratio of the (time-averaged, non-relativistic) power radiated: $\frac{P_{\text{bigger}}}{P_{\text{smaller}}} =$?

Problem 3

For the electric potential $V(\mathbf{r}, t) = \alpha \cos(\omega t)\, r$ (α and ω are constants), what is the charge density that generates it? Find an associated current density (assume $\mathbf{J}(\mathbf{r}, t = 0) = 0$).

Problem 4

Given a function ψ with $\Box\psi = 0$, show that $\mathbf{E} = \nabla \times (\mathbf{a}\,\psi)$ has $\Box\mathbf{E} = 0$ for constant vector $\mathbf{a}$, and find the associated vacuum magnetic field, $\mathbf{B}$, assuming $\mathbf{a} \cdot \nabla\psi = 0$. Write out the electromagnetic energy density and Poynting vector for these fields (in all cases, express your answers in terms of $\mathbf{a}$, ψ and its spatial/temporal derivative(s) as needed).