Problem Set 3

Physics 322
Electrodynamics II

Due on Friday, February 9th, 2024

Problem 1

A function $f(x)$ has a single root at $\bar{x}$, $f(\bar{x}) = 0$. Show that (assuming $f'(\bar{x}) \neq 0$)

$$\delta(f(x)) = \frac{\delta(x - \bar{x})}{|f'(x)|}. \tag{1}$$

You can do this by establishing that

$$\int_{-\infty}^{\infty} \delta(f(x)) p(x) dx = \frac{p(\bar{x})}{|f'(\bar{x})|}, \tag{2}$$

which is how equations like (1) are used in practice.

Problem 2

For a scalar function $f(x, y, z, t)$ in L, moving to the coordinate system $\bar{L}$ (which travels along a shared x axis at constant speed v) amounts to expressing x, y, z and t in terms of $\bar{x}$, $\bar{y}$, $\bar{z}$ and $\bar{t}$, and we call the resulting function $\bar{f}(\bar{x}, \bar{y}, \bar{z}, \bar{t})$.[1] Show that

$$-\frac{1}{c^2}\frac{\partial^2 \bar{f}}{\partial \bar{t}^2} + \frac{\partial^2 \bar{f}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{f}}{\partial \bar{y}^2} + \frac{\partial^2 \bar{f}}{\partial \bar{z}^2} = -\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}, \tag{3}$$

establishing that the D'Alembertian is a "scalar" (does not change under a Lorentz boost).

[1] Formally, then, we have

$$\bar{f}(\bar{x}, \bar{y}, \bar{z}, \bar{t}) = f(x(\bar{x}, \bar{y}, \bar{z}, \bar{t}), y(\bar{x}, \bar{y}, \bar{z}, \bar{t}), z(\bar{x}, \bar{y}, \bar{z}, \bar{t}), t(\bar{x}, \bar{y}, \bar{z}, \bar{t}))$$

Problem 3

The "four-gradient" ∂^μ (generalizing the three-dimensional ∇) acting on a scalar function $f(x, y, z, t)$ (defined in the previous problem) is defined by

$$\partial^\mu f \doteq \begin{pmatrix} -\frac{1}{c}\frac{\partial f}{\partial t} \\ \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} . \tag{4}$$

Show that $\partial^\mu f$ is a four-vector, i.e. that

$$\bar{\partial}^\mu \bar{f} = \sum_{\nu=0}^{3} \Lambda^\mu_\nu (\partial^\nu f) . \tag{5}$$

Problem 4

A charge density $\bar{\rho}$ is at rest in $\bar{L}$ (you might think of a model charge density: a total charge Q spread uniformly through a cube of side length $\bar{\ell}$), using Lorentz contraction, what are the values of ρ and $\mathbf{J}$ in L? Does the relation between $\{\rho c, \mathbf{J}\}$ and $\{\bar{\rho} c, \bar{\mathbf{J}}\}$ support the idea that ρ and $\mathbf{J}$ form a four vector?

Problem 5

Compute the potential $\bar{V}$ and vector potential $\bar{\mathbf{A}}$ for an infinite line of charge with uniform density $\bar{\lambda}$ lying at rest along the $\hat{\mathbf{z}}$ axis. Do the same thing if that line of charge is moving with speed v up the $\hat{\mathbf{z}}$ axis, i.e. find the potentials V and $\mathbf{A}$ for this configuration. Are the two sets of potentials related in the way you expect (i.e. do the components transform as a four-vector)? Compute the quadratic $-(V/c)^2 + \mathbf{A} \cdot \mathbf{A}$ in both cases, do the expressions agree?

Presentation Problems

Here are three presentation problems (see syllabus for a description of the writeup/in-class presentation). Everyone should solve these problems – if you choose to present one, either in-class or in written form, it will be due on Friday, February 9th (I'll ask for volunteers for in-class presentations on Monday, February 5th).

Problem 1*

For a particle that travels along the trajectory $\mathbf{w}(t)$ with $|\dot{\mathbf{w}}(t)| < c$ for all times, show that there is only one retarded time location, $\mathbf{w}(t_r)$ satisfying

$$c(t - t_r) = |\mathbf{r} - \mathbf{w}(t_r)|. \tag{6}$$

What happens if a particle travels faster than c? This situation can happen in materials, where the speed of light in material can be exceeded.

Problem 2*

A point charge sits at the origin of $\bar{L}$ that moves through L with speed v along a shared x axis. Write down the charge density in $\bar{L}$ and L separately, using the appropriate point charge density in each frame. For your charge density in $\bar{L}$, express $\bar{x}$, $\bar{y}$ and $\bar{z}$ in terms of the coordinates in L and use properties of the delta function to show that $\rho = \gamma\bar{\rho}$. Interestingly, then, length contraction occurs even for point charges . . .

Problem 3*

Positronium consists of an electron (with charge $-q$) and a positron (with charge $+q$) bound together. A classical model for positronium follows the Bohr model: The two charges attract one another, and move in a circle of radius R. Assuming the charges are always found as far apart as possible along their circular trajectory (so that, for example, if one is at $R/2\hat{\mathbf{x}}$, the other is at $-R/2\hat{\mathbf{x}}$), show that E&M predicts that positronium cannot exist unless you use both retarded and advanced interactions.