Problem Set 2

Physics 322
Electrodynamics II

Due on Friday, February 2nd, 2024

Problem 1

We saw that the "rapidity" had:

$$\cosh\eta = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \sinh\eta = \frac{v}{c}\cosh\eta \tag{1}$$

so that $\tanh\eta = v/c$. Write the velocity addition formula in terms of rapidities – use $\tanh\eta = v/c$, $\tanh\psi = \bar{v}/c$ and $\tanh\chi = v_L/c$ (where v is the speed of $\bar{L}$ relative to L, $\bar{v}$ is the speed of the "ball" thrown in $\bar{L}$, relative to $\bar{L}$, and v_L is the speed of the ball relative to L). Use hyperbolic trigonometric identities to establish that "rapidities add."

Problem 2

The velocity addition expression we worked out in class was for velocity components parallel to the "boost" direction. If, in $\bar{L}$, you throw a ball with speed $\bar{v}$ relative to $\bar{L}$ but in the $\hat{\mathbf{y}}$ direction, what is the velocity of the ball (both components) in L?

Problem 3

A particle moves along the $\hat{\mathbf{x}}$ axis with constant speed v_0 in the lab, so that the particle's location is given by $x(t) = v_0\,t$. Find the proper time τ of the particle in terms of t, and write $x(\tau)$, i.e. the position in the lab with proper time instead of coordinate time as the parameter.

Problem 4

A particle starts from rest, and accelerates to the right in a lab. It is moving relativistically, and has the following relation between proper time and coordinate time:

$$\tau(t) = \alpha \sinh^{-1}\left(\frac{t}{\alpha}\right) \tag{2}$$

where α is a constant with temporal dimension. Find the motion of the particle in the lab (i.e. what is $x(t)$?) – what is the force acting on this particle?

Problem 5

What is the Pythagorean distance travelled as a particle moves from $t = 0 \rightarrow T = 2\pi/\omega$ according to

$$\mathbf{r}(t) = R(\cos(\omega t)\hat{\mathbf{x}} + \sin(\omega t)\hat{\mathbf{y}})? \tag{3}$$

(the vector $\mathbf{r}(t)$ points from the origin to the location of the charge at time t). What is the associated Minkowski distance, obtained from the infinitesimal definition $ds^2 = -c^2 dt^2 + dx^2 + dy^2$?

Presentation Problems

Here are three presentation problems (see syllabus for a description of the writeup/in-class presentation). Everyone should solve these problems – if you choose to present one, either in-class or in written form, it will be due on Friday, February 2nd (I'll ask for volunteers for in-class presentations on Monday, January 29th).

Problem 1*

In the "twin paradox," two people of the same age start on earth, and one heads out into space – after a time t_f on earth, the space twin returns. The question is: how old is the space twin? As a concrete model, suppose the rocket travels according to $x(t) = a/2\,(1 - \cos(\omega\,t))$ – in terms of ω, what is t_f? Find the proper time τ_f elapsed in the rest frame of the rocket during this earth-based time-interval – that is the time elapsed for the space twin. The following integral identity will be useful:

$$\int_0^{2\,\pi} \sqrt{1 - k^2\,\sin^2\theta}\,d\theta = 4\,\mathsf{E}(k) \tag{4}$$

where the (complete) elliptic integral (of the second kind) on the right is defined by:

$$\mathsf{E}(k) \equiv \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2\theta}\, d\theta. \tag{5}$$

The maximum speed of the rocket twin is $v_{\text{max}} = a\,\omega/2$. Define $\beta_{\text{max}} \equiv v_{\text{max}}/c$, and make a plot of the total time elapsed on earth (t_f) and on the rocket (τ_f) in units of $1/\omega$ (so that the earth-based time is just $2\,\pi$) for $\beta_{\text{max}} = 0 \to 1$.

Problem 2*

A pair of infinite sheets of charge lie in the x–y plane – the top one has charge density $\sigma_0 > 0$ (when at rest) and the bottom one has $-\sigma_0$ (at rest). The top sheet moves along the $\hat{\mathbf{y}}$ axis with speed v, and the bottom sheet moves in the $-\hat{\mathbf{y}}$ direction with speed v.

a. Find the electric and magnetic fields above the top sheet.

b. Suppose you move along the $\hat{\mathbf{y}}$ axis with speed v – find the electric and magnetic fields in this moving reference frame (using the physical characteristics of the sheets that would be observed in this frame).

Problem 3*

Newtonian gravity looks structurally a lot like electrostatics: There is a gravitational field $\mathbf{g}$ (force per unit mass) with $\nabla \cdot \mathbf{g} = -4\pi G \rho_m$ where ρ_m is a mass density (mass per unit volume). Find the gravitational field associated with an infinite line of mass with mass density λ_0. What is the force of the field acting on a mass m a distance d away from the line of mass? What changes in the force if the line of mass is now moving to the right with speed v, and the mass m, still a distance d from the line source, is moving to the right with speed u?

Given our discussion of relativistic force transformation, what do you conclude about Newtonian gravity as a special relativistic theory?

Problem 4*

Make a Minkowski diagram (for a "lab" L, with coordinates x and ct) with an event at location x_0, time t_0. On your diagram, draw the axes associated with a frame $\bar{L}$ moving to the right through L with constant speed v, i.e. draw the $\bar{x}$ and $c\bar{t}$ axes. Show, geometrically, how you construct the event coordinates $\bar{x}_0$, $c\bar{t}_0$ with respect to these axes. How do the axes change as you increase v?