Problem Set 12

Physics 322
Electrodynamics II

Due on Friday, April 26th, 2024

Problem 1

I claimed that rotations and Lorentz boosts do not change the conservation law story much. But for the Lagrangian piece, this is not clear:

$$\mathcal{L}(x^\mu + M^\mu{}_\nu x^\nu) \approx \mathcal{L}(x^\mu) + \partial_\mu \mathcal{L} M^\mu{}_\nu x^\nu, \tag{1}$$

and in order to use the same technology that we had for a constant shift ($x^\mu \longrightarrow x^\mu + d^\mu$), we need:

$$\partial_\mu (\mathcal{L} M^\mu{}_\nu x^\nu) = (\partial_\mu \mathcal{L})\, M^\mu{}_\nu x^\nu. \tag{2}$$

Show that this is the case for both infinitesimal rotations and boosts (the "matrix" $M^\mu{}_\nu$ is what determines what type of infinitesimal transformation you have).

Problem 2

Identify the valid expressions written using Einstein summation notation:

$$\begin{aligned} &\text{i. } A^\mu B_\mu C^\nu = K^\nu \quad \text{ii. } A^\mu B_\mu C^\mu = K^\nu \quad \text{iii. } W^\mu = V^\nu \\ &\text{iv. } R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g_{\alpha\beta} R^{\alpha\beta} = \kappa T_{\mu\nu} \quad \text{v. } A^\mu B^\mu = S. \end{aligned} \tag{3}$$

For the invalid expressions, indicate what is wrong, and provide a fix (that process will not be unique).

Problem 3

An infinite line of charge with uniform charge density $\bar{\lambda}$ lies at rest along the $\hat{\mathbf{z}}$ axis in $\bar{L}$. What are the electric and magnetic fields in $\bar{L}$? Now suppose you have a lab L through which $\bar{L}$ is moving with speed v along a shared $\hat{\mathbf{z}}$ axis.

What are the electric and magnetic fields in L (calculate these directly using Physics 321 techniques – i.e. do not use the relativistic transformation of the fields)? Compute the invariants you found last week: $2(-E^2/c^2 + B^2)$ and $-4\mathbf{E} \cdot \mathbf{B}/c$ in both frames.

Problem 4

For an electric field $\mathbf{E}(t)$, the Fourier transform is (see notes from January 31st for a review of the Fourier transform conventions in this class)

$$\tilde{\mathbf{E}}(f) \equiv \int_{-\infty}^{\infty} \mathbf{E}(t) e^{i2\pi f t} dt. \tag{4}$$

You can use $\tilde{\mathbf{E}}(f)$ to find the frequency spectrum of radiation. Show that the following is true:

$$\int_{-\infty}^{\infty} E(t)^2 dt = \int_{-\infty}^{\infty} |\tilde{E}(f)|^2 df \tag{5}$$

assuming $\mathbf{E}(t)$ is real, and where $|\tilde{E}(f)|^2 \equiv \tilde{\mathbf{E}}(f)^* \cdot \tilde{\mathbf{E}}(f)$ (this result is needed in developing the angle resolved frequency spectrum).

Problem 5

A uniform magnetic field $\bar{B}$ points in the $\hat{\mathbf{z}}$ direction in $\bar{L}$. A charge q sits at rest in $\bar{L}$. Using the transformation of the field strength tensor, $F^{\mu\nu}$, what are the electric and magnetic fields in L, through which $\bar{L}$ moves with speed v along a shared $\hat{\mathbf{x}}$ axis, and what is the net force on the charge q in L?

Presentation Problems

Here are some presentation problems (see syllabus for a description of the writeup/in-class presentation). Everyone should solve these problems – if you choose to present one in written form, it will be due on Friday, April 26th. Everyone has completed an in-class presentation problem, but if you would like a "last hurrah" on the final day of class, you are welcome to present one of these – I'll see if there are any volunteers on Monday, April 22nd.

Problem 1*

The scalar field equation (wave equation) is:

$$-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}+\nabla^2\phi=0. \tag{6}$$

This field equation is unchanged under $\phi \to \phi+\kappa$ for constant κ. From the scalar field Lagrangian, $\mathcal{L}=\partial_\alpha\phi\partial^\alpha\phi/2$, what is the associated conserved J^μ (i.e. find J^μ using Noether's theorem), with $\partial_\mu J^\mu=0$?

For a true field theory, the sources must themselves be fields. If you have a conserved J^μ, you can couple it to E&M via a term of the form $pJ^\mu A_\mu$ (for coupling constant p) in the combined Lagrangian:

$$\mathcal{L}=\frac{1}{4\mu_0}F^{\alpha\beta}F_{\alpha\beta}+\frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi+pJ^\mu A_\mu. \tag{7}$$

Use the J^μ you found above to make this combined Lagrangian, and compute the field equations for ϕ and A^μ – they will now be coupled together.

Problem 2*

For the infinitesimal rotation matrices, $\mathbb{L}_x$, $\mathbb{L}_y$ and $\mathbb{L}_z$, we can recover the non-infinitesimal form by exponentiating, for example:

$$e^{\theta\mathbb{L}_z}=\mathbb{O}_z. \tag{8}$$

If you have a vector $\hat{\mathbf{n}}$, and you want to generate a rotation about the $\hat{\mathbf{n}}$ axis, the exponentiation of $\hat{\mathbf{n}}\cdot\mathbb{L}$ with $\mathbb{L}\equiv\mathbb{L}_x\hat{\mathbf{x}}+\mathbb{L}_y\hat{\mathbf{y}}+\mathbb{L}_z\hat{\mathbf{z}}$ will give you the appropriate rotation transformation matrix.

Working by analogy with rotations, find the Lorentz boost associated with an arbitrary velocity vector $\mathbf{v}$ (i.e. generate a boost in the $\hat{\mathbf{v}}$ direction with speed v).

Problem 3*

There is a nice review article on meta-materials:

John B. Pendry, David R. Smith, "Reversing Light with Negative Refraction," *Physics Today*, **57** (6), 37–43 (2004).

Study it, and summarize its content, focusing on the implications of a negative index of refraction.