Problem Set 11

Physics 322
Electrodynamics II

Due on Friday, April 19th, 2024

Problem 1

a. Render the following expression in summation notation:

$$A^{\mu\nu} = \sum_{\alpha,\beta=0}^{3} F^{\mu\alpha} b_\alpha G^{\nu\beta} f_\beta \text{ for } \mu, \nu = 0, 1, 2, 3. \tag{1}$$

b. Restore the summation symbol and "magic words" in the following:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g_{\alpha\beta} R^{\alpha\beta} = \kappa T_{\mu\nu}. \tag{2}$$

c. Show that $A^\mu B_\mu = A_\mu B^\mu$.

Problem 2

For $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, we need to lower indices to get $F_{\mu\nu}$. Use the Minkowski metric to lower indices and write out the components (in terms of the Cartesian components of $\mathbf{E}$ and $\mathbf{B}$) of

$$F_{\alpha\beta} = \eta_{\alpha\mu} \eta_{\beta\nu} F^{\mu\nu}. \tag{3}$$

Construct the scalar $F^{\mu\nu} F_{\mu\nu}$, express it in terms of $\mathbf{E}$ and $\mathbf{B}$.

Problem 3

The "dual field strength tensor," $G^{\mu\nu}$, is defined by: $G^{\mu\nu} = (1/2)\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$ where $\epsilon^{\mu\nu\alpha\beta}$ is the four-dimensional Levi-Civita symbol:

$$\epsilon^{\mu\nu\alpha\beta} \doteq \begin{cases} 1 & \text{if } \mu\nu\alpha\beta \text{ is an even permutation of } 0,1,2,3 \\ -1 & \text{if } \mu\nu\alpha\beta \text{ is an odd permutation of } 0,1,2,3 \\ 0 & \text{else.} \end{cases} \tag{4}$$

Write out the components of $G^{\mu\nu}$, and compute the scalars $F_{\mu\nu}G^{\mu\nu}$ and $G_{\mu\nu}G^{\mu\nu}$, write all expressions in terms of the Cartesian components of $\mathbf{E}$ and $\mathbf{B}$.

Problem 4

Show that a second rank tensor, $T_{\alpha\beta}$, can be written as a sum of a symmetric tensor $S_{\alpha\beta} = S_{\beta\alpha}$ and an antisymmetric tensor $A_{\alpha\beta} = -A_{\beta\alpha}$. Show that for antisymmetric $A_{\mu\nu}$ and symmetric $S^{\alpha\beta}$, $A_{\rho\sigma}S^{\rho\sigma} = 0$. Using these results, show that for a tensor $T^{\mu\nu}$ with symmetric piece $T_s^{\mu\nu}$ and antisymmetric piece $T_a^{\mu\nu}$, the scalar sum of $T^{\mu\nu}$ with an antisymmetric $A_{\mu\nu}$ is: $T^{\mu\nu}A_{\mu\nu} = T_a^{\mu\nu}A_{\mu\nu}$.

Problem 5

The scalar stress tensor is:

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \frac{1}{2}\eta^{\mu\nu}(\partial^\alpha\phi\partial_\alpha\phi)\,. \tag{5}$$

Show that for a scalar field (i.e. one that satisfies its field equation, $\partial_\mu\partial^\mu\phi = 0$), $\partial_\mu T^{\mu\nu} = 0$.

Problem 6

Under what conditions does $A^\mu = P^\mu e^{ik_\alpha x^\alpha}$ with constant P^μ and k_α solve

$$\Box A^\beta = 0 \text{ with } \partial_\gamma A^\gamma = 0? \tag{6}$$

Write your constraints on P^μ and k_α using indexed notation.

Presentation Problems

Here are some presentation problems (see syllabus for a description of the writeup/in-class presentation). Everyone should solve these problems – if you choose to present one, either in-class or in written form, it will be due on Friday, April 19th (I'll ask for volunteers for in-class presentations on Monday, April 15th).

Problem 1*

A Lorentz boost in the $\hat{\mathbf{x}}$ direction gives coordinates related by:

$$\begin{pmatrix} c\bar{t} \\ \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & -\frac{v}{c}\gamma & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\equiv \mathbb{B}_x} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}. \tag{7}$$

Write the infinitesimal form (for small v/c) of the matrix $\mathbb{B}_x$ appearing in the transformation, it should look like: $\mathbb{B}_x \approx \mathbb{I} + (v/c)\mathbb{K}_x$. By considering infinitesimal boosts in the y and z directions, identify the set $\mathbb{B}_x$, $\mathbb{B}_y$, $\mathbb{B}_z$, and associated $\mathbb{K}_x$, $\mathbb{K}_y$ and $\mathbb{K}_z$. Compute the commutators of the $\mathbb{K}$-matrices with each other. [1] Write the commutation relations in index form (i.e. $[\mathbb{K}_i, \mathbb{K}_j] =?$ for $i, j = 1, 2, 3$).

Problem 2*

The complex scalar field Lagrangian

$$\mathcal{L} = \partial_\mu \Phi^* \partial^\mu \Phi + \mu^2 \Phi^* \Phi \tag{8}$$

is unchanged by a complex rotation: $\Phi \to e^{i\theta}\Phi$ for constant θ. What is the associated conserved "charge" and current density?

Problem 3*

A rock skips across a pond – each skip produces ripples that travel with the characteristic speed of waves in the water, call it v. Suppose the rock moves with a speed $v_r > v$, sketch the ripple patterns as time goes on. Show (graphically) that there is a "wedge" of ripples behind the rock (think of the wake of a boat) and find the angle of that wedge. What are the implications of motion faster than the local wave propagation speed for calculations of, for example, the retarded time?

[1] A commutator of two matrices is the difference: $[\mathbb{A}, \mathbb{B}] \equiv \mathbb{A}\mathbb{B} - \mathbb{B}\mathbb{A}$.