Problem Set 10

Physics 322
Electrodynamics II

Due on Friday, April 12th, 2024

Problem 1

a. In a frame L, we have $V = 0$ and $\mathbf{A} = P_0 \cos(kx - \omega t)\hat{\mathbf{y}}$. Find the electric and magnetic fields in L, and compute the intensity $\mathbf{I}$.

b. The four-potential, A^μ is a four-vector. Find the components of the potential from part a. in a frame $\bar{L}$ moving at speed v to the right through L along a shared x axis. From the potentials, find the electric and magnetic fields, identify the $\bar{k}$ and $\bar{\omega}$ in $\bar{L}$, show that the wave speed remains c, and find the intensity. What is the frequency of the wave, $\bar{\omega}$, in $\bar{L}$ (i.e. how is $\bar{\omega}$ related to ω)?

c. What happens to the field amplitudes, wave frequency and intensity as $v \to c$?

Problem 2

You showed that the "four-gradient":

$$\partial^\mu \equiv \frac{\partial}{\partial x^\mu} \doteq \begin{pmatrix} -\frac{1}{c}\frac{\partial}{\partial t} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \tag{1}$$

is a vector under a Lorentz transformation. Using this operator, and the four-potential A^μ,

$$A^\mu \doteq \begin{pmatrix} V/c \\ A^x \\ A^y \\ A^z \end{pmatrix}, \tag{2}$$

generate the content of:

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \quad \mathbf{B} = \nabla \times \mathbf{A} \tag{3}$$

using only the building blocks ∂^μ and A^ν (i.e. write an equation that has the components of $\mathbf{E}$ and $\mathbf{B}$ on one side, and the correct derivatives of the potential on the other).

Problem 3

Suppose you had an action of the form:

$$S[x(t)] = \int_0^T L(x(t), \dot{x}(t), \ddot{x}(t))dt. \tag{4}$$

What are the "equations of motion" here if you have fixed (given) endpoints for both $x(t)$ and $\dot{x}(t)$? Work this out carefully, as we did in class. Can you "guess" what the equation of motion looks like if L also includes dependence on $\dddot{x}(t)$?

Problem 4

a. For the Lagrangian:

$$L = \frac{1}{2}m\dot{x}(t)^2 - U(x(t)) + F(x(t))\dot{x}(t), \tag{5}$$

where $F(x(t))$ is some function of $x(t)$, what is the resulting equation of motion for $x(t)$?

b. Is it possible to add a function to the Lagrangian $L = 1/2m\dot{x}(t)^2 - U(x(t))$ in order to get an equation of motion that includes the radiation reaction force:

$$m\ddot{x}(t) = -\frac{dU}{dx} - m\tau\dddot{x}(t)? \tag{6}$$

Presentation Problems

Here are some presentation problems (see syllabus for a description of the writeup/in-class presentation). Everyone should solve these problems – if you choose to present one, either in-class or in written form, it will be due on Friday, April 12th (I'll ask for volunteers for in-class presentations on Monday, April 8th).

Problem 1*

For points in the xy plane, a rotation about the z axis defines new coordinates:

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\equiv \mathbb{O}_z} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{7}$$

Make a sketch of the coordinate axes, is this a clockwise or counterclockwise rotation of the axes?

Write the infinitesimal form (for small θ) of the matrix $\mathbb{O}_z$ appearing in the transformation, it should look like: $\mathbb{O}_z \approx \mathbb{I} + \theta \mathbb{L}_z$. By considering infinitesimal rotations about the x and y axes (keeping the sense of the rotation the same), identify the set $\mathbb{O}_x$, $\mathbb{O}_y$, $\mathbb{O}_z$, and associated $\mathbb{L}_x$, $\mathbb{L}_y$ and $\mathbb{L}_z$. Compute the commutators of the $\mathbb{L}$-matrices with each other. [1] Write the commutation relations in index form (i.e. $[\mathbb{L}_i, \mathbb{L}_j] = ?$ for $i, j = 1, 2, 3$) using the Levi-Civita symbol.

Problem 2*

A square loop of wire (side length $\bar{a}$) carrying steady, neutral current $\bar{I}$ sits at rest in $\bar{L}$ (moving to the right with speed v through a frame L). The loop's magnetic dipole moment has magnitude $\bar{m} = \bar{I}\bar{a}^2$ and its electric dipole moment is zero. What is the electric dipole moment of the loop in L?

Problem 3*

For the Lagrangian:

$$L = \frac{1}{2} m v^2 - qV + q\mathbf{v} \cdot \mathbf{A}, \tag{8}$$

show that the equations of motion correctly return: $m\ddot{\mathbf{r}}(t) = q\mathbf{E} + q\dot{\mathbf{r}} \times \mathbf{B}$.

[1] A commutator of two matrices is the difference: $[\mathbb{A}, \mathbb{B}] \equiv \mathbb{A}\mathbb{B} - \mathbb{B}\mathbb{A}$.