Scattering

J. Franklin

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1 Scattering Setup

An electromagnetic plane wave is incident on a material. The charges in the material begin to move under the influence of the incident wave, and the acceleration of the charges generates electromagnetic fields. Question: By measuring the resulting fields, from afar, what can we say about the nature of the material?

In the simplest configuration, an electromagnetic plane wave travels along the z axis, and we'll take the polarization to be $\hat{\mathbf{x}}$ as shown in the figure below. The (real) fields are

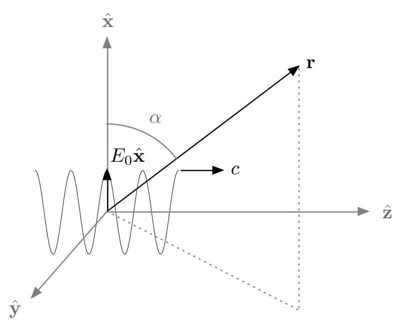

Figure 1: A plane wave travels along the z axis, polarized in the x direction.

$$\mathbf{E} = E_0 \cos(k(z - ct))\hat{\mathbf{x}} \qquad \mathbf{B} = \frac{E_0}{c} \cos(k(z - ct))\hat{\mathbf{y}}. \tag{1}$$

These fields act on a charge q (with mass m) initially at rest at the origin. The equations of motion

$$m\ddot{\mathbf{r}}(t) = q\left(\mathbf{E} + \dot{\mathbf{r}}(t) \times \mathbf{B}\right) \tag{2}$$

become, in Cartesian coordinates,

$$\begin{aligned} m\ddot{x}(t) &= qE_0 \cos(k(z(t) - ct)) - qE_0 \frac{\dot{z}(t)}{c} \cos(k(z(t) - ct)) \\ m\ddot{y}(t) &= 0 \\ m\ddot{z}(t) &= qE_0 \frac{\dot{x}(t)}{c} \cos(k(z(t) - ct)). \end{aligned} \tag{3}$$

We're already in trouble — except for $y(t)=0$, we cannot easily obtain solutions to the coupled, nonlinear ODEs we are presented with.

So we begin making assumptions about the motion. Initially, the charge is at rest, and we see that the magnetic component of the force goes like v/c, so that as long as the charge doesn't start moving at relativistic speeds, the dominant force contribution will be from the electric field. If we remove the magnetic force, then we can solve $\ddot{z}(t)=0$ with the given initial conditions, $z(t)=0$, and the equation of motion for $x(t)$ is solvable by quadrature,

$$x(t)=\frac{qE_0}{mk^2c^2}\left(1-\cos(kct)\right). \tag{4}$$

What does our "non-relativistic" assumption mean here? We have

$$\frac{\dot{x}(t)}{c}=\frac{qE_0}{mkc^2}\sin(kct), \tag{5}$$

which has a maximum value $(qE_0/k)/(mc^2)$. If we define the length $d\equiv(qE_0/k^2)/(mc^2)$, then our assumption that the velocity is small amounts to $d\ll 1/k$, or, using the wavelength of the original incident plane wave, $\lambda=2\pi/k$, $d\ll\lambda$.

Given the solution (4), we have an oscillating dipole $p(t)=qx(t)$, which itself generates the radiation fields

$$\begin{aligned}
\mathbf{B}_d &= -\frac{\mu_0}{4\pi rc}\left(\hat{\mathbf{r}}\times\ddot{\mathbf{p}}(t_r)\right)\\
&= -\frac{\mu_0 q\ddot{x}(t_r)}{4\pi rc}\left(\sin\phi\hat{\boldsymbol{\theta}}+\cos\theta\cos\phi\hat{\boldsymbol{\phi}}\right)\\
\mathbf{E}_d &= -c\hat{\mathbf{r}}\times\mathbf{B}_d\\
&= \frac{\mu_0 q\ddot{x}(t_r)}{4\pi r}\left(-\cos\theta\cos\phi\hat{\boldsymbol{\theta}}+\sin\phi\hat{\boldsymbol{\phi}}\right).
\end{aligned} \tag{6}$$

with $t_r\equiv t-r/c$. Putting in the expression for the acceleration,

$$\begin{aligned}
\mathbf{E}_d &= \frac{\mu_0 q^2 E_0}{4\pi mr}\cos(kct_r)\left(-\cos\theta\cos\phi\hat{\boldsymbol{\theta}}+\sin\phi\hat{\boldsymbol{\phi}}\right)\\
\mathbf{B}_d &= -\frac{\mu_0 q^2 E_0}{4\pi mrc}\cos(kct_r)\left(\sin\phi\hat{\boldsymbol{\theta}}+\cos\theta\cos\phi\hat{\boldsymbol{\phi}}\right).
\end{aligned} \tag{7}$$

The quantity $\mu_0 q^2/(4\pi m)\equiv\ell$ defines a length scale (the "classical electron radius," if the oscillating particle is an electron). In terms of this length scale, the Poynting vector is

$$\mathbf{S}_d=\frac{1}{\mu_0}\mathbf{E}_d\times\mathbf{B}_d=\frac{1}{\mu_0 c}E_0^2\frac{\ell^2}{r^2}\cos^2(kc(t-r/c))\left(\cos^2\theta\cos^2\phi+\sin^2\phi\right)\hat{\mathbf{r}}. \tag{8}$$

We can average over one full cycle of the dipole radiation to get the intensity, which just picks up a factor of $\frac{1}{2}$ as usual,

$$\mathbf{I}_d=\langle\mathbf{S}\rangle=\frac{1}{2}\left(\frac{1}{\mu_0 cr^2}E_0^2\ell^2\right)\left(\cos^2\theta\cos^2\phi+\sin^2\phi\right)\hat{\mathbf{r}}. \tag{9}$$

The incoming plane wave has intensity

$$\mathbf{I}_0 = \frac{1}{2}\frac{1}{\mu_0 c}E_0^2\hat{\mathbf{z}} \tag{10}$$

and we can write the dipole intensity in terms of the magnitude I_0:

$$\mathbf{I}_d = I_0\frac{\ell^2}{r^2}\left(\cos^2\theta\cos^2\phi + \sin^2\phi\right)\hat{\mathbf{r}}. \tag{11}$$

As a final note, this intensity is often quoted with the angle between the dipole vector and the field point. Call that angle α, then for our $\hat{\mathbf{x}}$-directed dipole moment, $\cos\alpha = \hat{\mathbf{x}}\cdot\hat{\mathbf{r}} = \sin\theta\cos\phi$ and then $\sin^2\alpha = \cos^2\theta\cos^2\phi + \sin^2\phi$, giving

$$\mathbf{I}_d = I_0\frac{\ell^2}{r^2}\sin^2\alpha\hat{\mathbf{r}}. \tag{12}$$

1.1 Scattering Cross-Section

In order to find the power radiated by the oscillating charge, we compute the flux of the intensity through a sphere of radius R,

$$P = \int_{\partial\Omega}\mathbf{I}_d\cdot d\mathbf{a} = \int_0^{2\pi}\int_0^{\pi} I_d R^2\sin\theta d\theta d\phi. \tag{13}$$

In order to capture the geometry of the radiation passing through the sphere, we use a short-hand for the angular elements, $d\Omega = \sin\theta d\theta d\phi$, known as the "solid angle," and then we can quote the angular distribution of the intensity using the notation,

$$\frac{dP}{d\Omega} = I_d R^2 = I_0\ell^2\left(\cos^2\theta\cos^2\phi + \sin^2\phi\right). \tag{14}$$

Be warned: this "differential cross section" notation is generally meant to stand-in for the integrand of a flux integral, as in (13). It does not indicate that we have a function $P(\Omega)$ whose derivative, w.r.t. Ω is of interest.

The total power can be computed, with the angular integrals throwing in an overall $2\pi 4/3$,

$$P = \frac{8\pi}{3}\ell^2 I_0. \tag{15}$$

1.2 General Polarization

The dipole moment of the charge above was in the $\hat{\mathbf{x}}$ direction because that was the electric field's direction. But we can take any polarization for the plane wave, as long as it is perpendicular to the propagation direction. Let that remain z, and this time, we'll let the polarization vector point anywhere in the xy plane:

$$\mathbf{E} = E_0\cos(k(z-ct))\underbrace{\left[\cos\psi\hat{\mathbf{x}} + \sin\psi\hat{\mathbf{y}}\right]}_{\equiv\hat{\mathbf{n}}} \tag{16}$$

where ψ is the angle made by $\mathbf{E}$ with respect to the $\hat{\mathbf{x}}$ axis.

Again, assuming the magnetic field's contribution is negligible, the dipole oscillation of the charge is now

$$\mathbf{p}(t) = \frac{q^2 E_0}{mk^2c^2}\cos(kct_r)\hat{\mathbf{n}} \tag{17}$$

with

$$\begin{aligned} \mathbf{B}_d &= -\frac{E_0}{c}\frac{\ell}{r}\cos(kct_r)\hat{\mathbf{r}} \times \hat{\mathbf{n}} \\ \mathbf{E}_d &= E_0\frac{\ell}{r}\cos(kct_r)\left(\hat{\mathbf{r}} \times \hat{\mathbf{n}}\right) \times \hat{\mathbf{r}}. \end{aligned} \tag{18}$$

The Poynting vector, written in terms of α, the angle between the dipole vector and the field point, is

$$\mathbf{S}_d = \frac{1}{\mu_0 c}E_0^2\frac{\ell^2}{r^2}\cos^2(kc(t - r/c))\sin^2\alpha\hat{\mathbf{r}}. \tag{19}$$

and then the intensity vector is

$$\mathbf{I}_d = \frac{I_0\ell^2}{r^2}\sin^2\alpha\hat{\mathbf{r}}. \tag{20}$$

In a typical "scattering" setup, we write our results in terms of the "scattering angle," the angle between the incident wave direction and the field point $\mathbf{r}$. Since we have aligned the direction of propagation with the z axis, the scattering angle is the polar θ. To recover the spherical coordinates from the angle between the polarization vector and the field point, we note that $\cos\alpha = \hat{\mathbf{n}} \cdot \hat{\mathbf{r}} = \sin\theta\cos(\phi - \psi)$, so that

$$\sin^2\alpha = 1 - \left(\hat{\mathbf{n}} \cdot \hat{\mathbf{r}}\right)^2 = 1 - \sin^2\theta\cos^2(\phi - \psi) \tag{21}$$

with

$$\mathbf{I}_d = \frac{I_0\ell^2}{r^2}\left(1 - \sin^2\theta\cos^2(\phi - \psi)\right)\hat{\mathbf{r}}. \tag{22}$$

The scattering of incident electromagnetic radiation with a free electron, which is what we have developed here, is called "Thompson scattering," and it is standard to average the intensity over all available polarizations,

$$\bar{\mathbf{I}}_d \equiv \frac{1}{2\pi}\int_0^{2\pi}\mathbf{I}_d d\psi = \frac{I_0\ell^2}{2r^2}\left(1 + \cos^2\theta\right)\hat{\mathbf{r}}. \tag{23}$$

Thompson scattering has the special property that its intensity is independent of the frequency ($\omega = kc$) of the incoming radiation. This is surprising from the point of view of an oscillating dipole, $p(t) = p_0\cos(\omega t)$ which has ω^4 in its Poynting vector ($\ddot{p}(t)^2$). What happened here? How did we lose the frequency dependence of the dipole oscillation?

2 Lorentz Model

Thompson scattering occurs when a free electron moves under the influence of electromagnetic waves. For electrons that are attached to atoms or molecules by forces that we model as springs, we have a damped, driven harmonic oscillator once the electromagnetic radiation impinges on the material.

Let's again take a wave that is traveling along the z axis, polarized, as above, in the xy plane with $\hat{\mathbf{n}} = \cos\psi\hat{\mathbf{x}} + \sin\psi\hat{\mathbf{y}}$. We'll again drop the magnetic force, under the assumption that $v/c \ll 1$ for the particle motion. Then, in the direction of the polarization vector, the equation of motion reads

$$m\ddot{u}(t) = -m\omega_0^2 u(t) - m\tau\dddot{u}(t) + qE_0 e^{ik(z-ct))} \tag{24}$$

where ω_0 is the angular frequency of the "charge oscillation," τ is the timescale associated with radiation damping ($\tau \equiv \mu_0 q^2/(6\pi mc)$), and the last term is the driving force provided by the plane wave. We have switched over to a complex exponential description of both the driving force and the position $u(t)$ — at the end of our calculation, we'll take the real part.

As in the Thompson scattering case, we know that $\ddot{z}(t) = 0$, which gives us $z(t) = 0$ once the initial conditions are in place. Then letting $kc \equiv \omega_e$, we have

$$\ddot{u}(t) = -\omega_0^2 u(t) - \tau\dddot{u}(t) + \frac{qE_0}{m}e^{-i\omega_e t}. \tag{25}$$

Now if we make the ansatz $u(t) = u_0 e^{-i\omega_e t}$, we find

$$u_0 = \frac{qE_0}{m\left(\omega_0^2 - \omega_e^2 - i\omega_e^3\tau\right)}. \tag{26}$$

x Let the damping factor $\gamma \equiv \omega_e^2\tau$ to make this damped, driven harmonic oscillator solution look more familiar. The steady state solution is

$$u(t) = \frac{qE_0}{m\left(\omega_0^2 - \omega_e^2 - i\gamma\omega_e\right)}e^{-i\omega_e t} = \frac{qE_0}{m}\frac{e^{-i\omega_e t + i\phi}}{\sqrt{\left(\omega_0^2 - \omega_e^2\right)^2 + \gamma^2\omega_e^2}} \tag{27}$$

with $\phi \equiv \tan^{-1}(\gamma\omega_e/(\omega_0^2 - \omega_e^2))$.

Putting it all together, the steady state motion of the particle, is

$$\mathbf{r}(t) = \frac{qE_0\cos(\omega_e t + \phi)\hat{\mathbf{n}}}{m\sqrt{\left(\omega_0^2 - \omega_e^2\right)^2 + \gamma^2\omega_e^2}}. \tag{28}$$

For the Poynting vector, we have

$$\mathbf{S}_d = \frac{1}{\mu_0 c}E_0^2\frac{\ell^2}{r^2}\frac{\omega_e^4}{\left(\omega_0^2 - \omega_e^2\right)^2 + \gamma^2\omega_e^2}\cos^2(\omega_e t + \phi)\sin^2\alpha\hat{\mathbf{r}}, \tag{29}$$

with α the angle between $\hat{\mathbf{n}}$ and $\hat{\mathbf{r}}$ as usual.

We can carry out the time-averaging, as usual, and write the resulting intensity in terms of the incident intensity, I_0, as we have done previously,

$$\mathbf{I}_d = \frac{I_0 \ell^2}{r^2} \frac{\omega_e^4}{\left(\omega_0^2 - \omega_e^2\right)^2 + \gamma^2 \omega_e^2} \sin^2 \alpha \hat{\mathbf{r}}. \tag{30}$$

Putting the spherical coordinates back in so that we can look at the power distribution, and returning $\gamma \to \omega_e^2 \tau$,

$$\mathbf{I}_d = \frac{I_0 \ell^2}{r^2} \frac{\omega_e^4}{\left(\omega_0^2 - \omega_e^2\right)^2 + \tau^2 \omega_e^6} \left(1 - \sin^2 \theta \cos^2(\phi - \psi)\right) \hat{\mathbf{r}}, \tag{31}$$

and averaging over polarization angle ψ,

$$\bar{\mathbf{I}}_d = \frac{I_0 \ell^2}{2r^2} \frac{\omega_e^4}{\left(\omega_0^2 - \omega_e^2\right)^2 + \tau^2 \omega_e^6} \left(1 + \cos^2 \theta\right) \hat{\mathbf{r}}. \tag{32}$$

For $\omega_0 = 0$ and $\tau = 0$ (no damping, and neglecting radiation reaction), we recover the Thompson result. The total power radiated, as a function of ω_e, is

$$P = \left(\frac{8\pi}{3} I_0 \ell^2\right) \frac{\omega_e^4}{(\omega_0^2 - \omega_e^2)^2 + \tau^2 \omega_e^6)}. \tag{33}$$