Noether's Theorem for Scalars

J. Franklin

April 15th, 2024

1 Setup

We will be considering a field $\phi(t, x, y, z)$, a function of $\{t, x, y, z\}$ (although I'll almost always suppress the coordinate dependence to avoid clutter). I'll be very explicit in developing the conserved current, to highlight the role of Taylor expansion and the use of the field equations, and then revert to some summation notation to economize the expressions, but there is no promise (as is often implicit in the notation) that the expressions we develop transform in any specific way — in the end, the structure of the Lagrangian determines the covariant nature of the field equations: If the field Lagrangian is a scalar under a rotation, then the field equations have rotational covariance, which means that the field equations retain their form in the new, rotated, coordinate system. Concretely, rotational covariance means that if the field equation is $A(x) = B(x)$ in the original coordinate system, then in the rotated coordinate system, it reads $\bar{A}(\bar{x}) = \bar{B}(\bar{x})$. Similarly, if the Lagrangian is a scalar under a Lorentz boost, then the field equations exhibit Lorentz covariance.

Given a Lagrangian $\mathcal{L}(\phi, \partial\phi)$ for ϕ, that depends on ϕ and its derivative, the field equation is:

$$-\left[\frac{1}{c}\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\frac{1}{c}\partial\left(\frac{\partial\phi}{\partial t}\right)} + \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial x}\right)} + \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial y}\right)} + \frac{\partial}{\partial z}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial z}\right)}\right] + \frac{\partial \mathcal{L}}{\partial\phi} = 0. \tag{1}$$

Define the operator ∂_μ to be

$$\partial_\mu \doteq \begin{pmatrix} \frac{1}{c}\frac{\partial}{\partial t} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}, \tag{2}$$

just a collection of derivatives at this stage.

Suppose we make a perturbation to the field $\phi \to \phi + \theta$ for θ small, induced in some manner. If the field equations are unchanged by this transformation, then the Lagrangian must also be (basically) unchanged. We allow a total divergence at most — for a set of functions $\{K^\mu\}_{\mu=0}^3$, the statement that the Lagrangian evaluated at $\phi+\theta$ should differ from the Lagrangian evaluated at ϕ by a "total divergence" reads

$$\mathcal{L}(\phi+\theta)=\mathcal{L}(\phi)+\frac{1}{c}\frac{\partial K^0}{\partial t}+\frac{\partial K^1}{\partial x}+\frac{\partial K^2}{\partial y}+\frac{\partial K^3}{\partial z}, \tag{3}$$

and if we put the Lagrangian under a volume integral to form an action, we can use the divergence theorem in four dimensions to turn the derivative terms into an integral over the surface of the domain, which then does not contribute to the variation w.r.t. ϕ.

We can make use of the ∂_μ notation to write the divergence compactly,

$$\frac{1}{c}\frac{\partial K^0}{\partial t}+\frac{\partial K^1}{\partial x}+\frac{\partial K^2}{\partial y}+\frac{\partial K^3}{\partial z}=\sum_{\mu=0}^{3}\partial_\mu K^\mu\equiv\partial_\mu K^\mu, \tag{4}$$

employing implicit summation over up-down pairs to remove the summation symbol. Using this summation convention (carefully!), the sums in the field equation (1) can be written compactly, as

$$-\partial_\mu\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}+\frac{\partial\mathcal{L}}{\partial\phi}=0. \tag{5}$$

Back to our field transformation, $\phi\to\phi+\theta$, we put this in to $\mathcal{L}$ and perform a Taylor expansion,

$$\begin{aligned}\mathcal{L}(\phi+\theta,\partial\phi+\partial\theta)&=\mathcal{L}(\phi,\partial\phi)+\frac{\partial\mathcal{L}}{\partial\phi}\theta+\frac{\partial\mathcal{L}}{\frac{1}{c}\partial\left(\frac{\partial\phi}{\partial t}\right)}\frac{1}{c}\frac{\partial\theta}{\partial t}+\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial x}\right)}\frac{\partial\theta}{\partial x}+\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial y}\right)}\frac{\partial\theta}{\partial y}+\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial z}\right)}\frac{\partial\theta}{\partial z}\\&=\mathcal{L}(\phi,\partial\phi)+\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\partial_\mu\theta,\end{aligned} \tag{6}$$

"compactifying" the sums in the second line.

In order to use the field equation information, we'll rewrite right-hand side of (6) using the product rule

$$\begin{aligned}\mathcal{L}(\phi+\theta,\partial\phi+\partial\theta)&=\mathcal{L}(\phi,\partial\phi)+\frac{\partial\mathcal{L}}{\partial\phi}\theta+\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\theta\right)-\theta\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\right)\\&=\mathcal{L}(\phi,\partial\phi)+\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\theta\right).\end{aligned} \tag{7}$$

Our requirement is that

$$\mathcal{L}(\phi+\theta,\partial\phi+\partial\theta)-\mathcal{L}(\phi,\partial\phi)=\partial_\mu K^\mu, \tag{8}$$

for some K^μ, or

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\theta-K^\mu\right)=0. \tag{9}$$

The conserved four-vector is then

$$J^\mu=\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\theta-K^\mu \tag{10}$$

with $\partial_\mu J^\mu=0$.

2 Example — Space-Time Translation

If we take $\bar{x}^\mu = x^\mu + d^\mu$ for constant d^μ, then

$$\phi(\bar{x}^\mu) = \phi(x^\mu + d^\mu) = \phi + d^\mu \partial_\mu \phi \tag{11}$$

so that $\theta = d^\mu \partial_\mu \phi$. The Lagrangian itself has

$$\mathcal{L}(\bar{x}^\mu) = \mathcal{L}(x^\mu + d^\mu) \approx \mathcal{L}(x) + d^\mu \partial_\mu \mathcal{L} = \mathcal{L}(x) + \partial_\mu \left(d^\mu \mathcal{L}\right), \tag{12}$$

and then $\mathcal{L}(\bar{x}) - \mathcal{L}(x) = \partial_\mu K^\mu$ gives $K^\mu = d^\mu \mathcal{L}$.

The conserved J^μ, from (10), is

$$J^\mu = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)} d^\nu \partial_\nu \phi - d^\nu \delta^\mu_\nu \mathcal{L} = d^\nu \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} \right) = d^\nu T^\mu_{\ \nu}, \tag{13}$$

and since d^ν is constant, we have not only $\partial_\mu J^\mu = 0$, but *also* $\partial_\mu T^\mu_{\ \nu} = 0$.

The object

$$T^\mu_{\ \nu} \equiv \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)} \phi_{,\nu} - \delta^\mu_\nu \mathcal{L} \tag{14}$$

is the stress-energy tensor. It is conserved provided the field equations are unchanged under a constant shift of coordinate. That conservation takes the form of:

$$\partial_\mu T^\mu_{\ \nu} = 0 = \frac{1}{c} \frac{\partial T^0_{\ \nu}}{\partial t} + \frac{\partial T^j_{\ \nu}}{\partial x^j} \tag{15}$$

where the Roman indices are summed from one to three: $j = 1, 2, 3$. Of course, this is still a set of four equations, for $\nu = 0$ and $\nu = k = 1, 2, 3$,

$$\frac{1}{c} \frac{\partial T^0_{\ 0}}{\partial t} + \frac{\partial T^j_{\ 0}}{\partial x^j} = 0 \qquad \frac{1}{c} \frac{\partial T^0_{\ k}}{\partial t} + \frac{\partial T^j_{\ k}}{\partial x^j} = 0. \tag{16}$$

3 Example — Infinitesimal Transformation

So far, we have only considered constant coordinate translations, but we can also make coordinate transformations that are functions of the coordinates themselves (rotations and boosts are examples, but so are most "actual" coordinate transformations, like between Cartesian and spherical).

This time let $\bar{x}^\mu = x^\mu + f^\mu$ where f^μ is a function of position, $f^\mu(x)$ that we'll take to be small. The example above ("Space-Time Translation") holds all the way down, we have

$$J^\mu = f^\nu T^\mu_{\ \nu} \text{ with } \partial_\mu J^\mu = 0. \tag{17}$$

This time, we have to be careful about pulling f^ν out, since it is not a constant. Suppose the transformation is linear (and, by assumption above, infinitesimal), $f^\nu = M^\nu_{\ \sigma} x^\sigma$. Then

$$J^\mu = M^\nu_{\ \sigma} x^\sigma T^\mu_{\ \nu} = M_{\nu\sigma} \left(x^\sigma T^{\mu\nu}\right). \tag{18}$$

For rotations, the infinitesimal transformation "matrix" $M_{\nu\sigma}$ is antisymmetric: $M_{\nu\sigma} = -M_{\sigma\nu}$, so that we only need to take the antisymmetric part of the tensor $x^\sigma T^{\mu\nu}$, it alone survives contraction with $M_{\nu\sigma}$,

$$J^\mu = \frac{1}{2} M_{\nu\sigma} \left(x^\sigma T^{\mu\nu} - x^\nu T^{\mu\sigma} \right). \tag{19}$$

Now we have isolated the constant "matrix" $M_{\nu\sigma}$, so we can slip it past the divergence in $\partial_\mu J^\mu = 0$:

$$\partial_\mu J^\mu = \frac{1}{2} M_{\nu\sigma} \partial_\mu \left(x^\sigma T^{\mu\nu} - x^\nu T^{\mu\sigma} \right), \tag{20}$$

and we have identified the conserved quantity

$$U^{\sigma\mu\nu} \equiv \left(x^\sigma T^{\mu\nu} - x^\nu T^{\mu\sigma} \right) \text{ with } \partial_\mu U^{\sigma\mu\nu} = 0. \tag{21}$$

4 Example — Real Scalar Fields

Take $\mathcal{L} = \frac{1}{2} \partial_\alpha \phi \partial^\alpha \phi$, with field equation $\Box \phi = 0$. The stress tensor becomes

$$T^\mu_\nu = \partial^\mu \phi \partial_\nu \phi - \frac{1}{2} \delta^\mu_\nu \partial_\alpha \phi \partial^\alpha \phi, \tag{22}$$

or in its upper form (raising the ν index using the metric),

$$T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - \frac{1}{2} \eta^{\mu\nu} \partial_\alpha \phi \partial^\alpha \phi. \tag{23}$$

This is conserved when $\Box \phi = 0$.

The Lorentz-transformation conserved $U^{\sigma\mu\nu}$ is

$$U^{\sigma\mu\nu} = \partial^\mu \phi \left(x^\sigma \partial^\nu \phi - x^\nu \partial^\sigma \phi \right) - \frac{1}{2} \partial_\alpha \phi \partial^\alpha \phi \left(x^\sigma g^{\mu\nu} - x^\nu g^{\mu\sigma} \right). \tag{24}$$

This is also conserved (harder to check than the above, but it does work out).