The Problem
Analysis
Adding Electric Field

Particle Motion in a Magnetic Field [1]

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Outline

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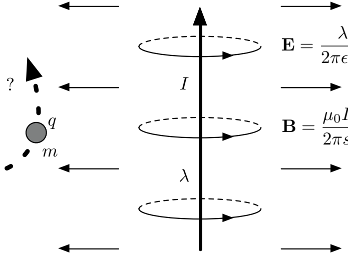
The Problem

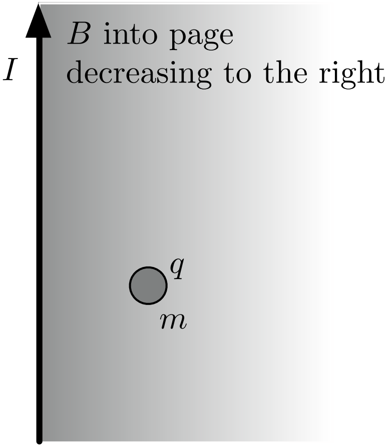
An infinite line carrying steady current *I* and uniform charge density λ lies along the $\hat{\mathbf{z}}$ axis. Question: How does a particle of charge *q*, mass *m*, move under the influence of the resulting electric and magnetic fields?



$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}} \qquad V = -\frac{\lambda}{2\pi\epsilon_0}\log\left(\frac{s}{s_0}\right)$$

$$\mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}} \qquad \mathbf{A} = -\frac{\mu_0 I}{2\pi}\log\left(\frac{s}{s_0}\right)\hat{\mathbf{z}}$$

We'll focus on the purely magnetic, $\lambda = 0$, case first. Any predictions?



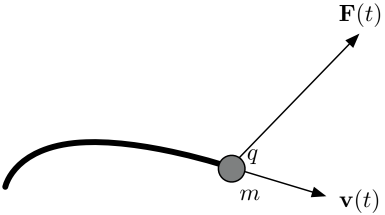
The equation of motion,

$$m\mathbf{a}(t) = q\mathbf{v}(t) \times \mathbf{B}(\mathbf{x}(t))$$

cannot be easily solved in closed form. So consider a numerical method like “Velocity Verlet":

$$\mathbf{x}(t + \Delta t) = \mathbf{x}(t) + \mathbf{v}(t)\Delta t + \frac{1}{2m}\mathbf{F}(t)\Delta t^2,$$

$$\mathbf{v}(t + \Delta t) = \mathbf{v}(t) + \frac{1}{2m}\left[\mathbf{F}(t) + \mathbf{F}(t + \Delta t)\right]\Delta t.$$



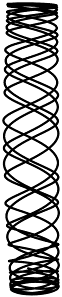
$$\mathbf{x}(t+\Delta t) = \mathbf{x}(t) + \mathbf{v}(t)\Delta t + \frac{1}{2m}\mathbf{F}(t)\Delta t^2,$$

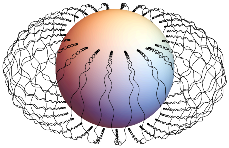
$$\mathbf{v}(t+\Delta t) = \mathbf{v}(t) + \frac{1}{2m}\left[\mathbf{F}(t) + \mathbf{F}(t+\Delta t)\right]\Delta t$$

If $\mathbf{F}(t)$ depends on position only: $\mathbf{F}(t) \equiv \mathbf{F}(\mathbf{x}(t))$ then we can evaluate the update: $\mathbf{F}(t+\Delta t) = \mathbf{F}(\mathbf{x}(t+\Delta t))$. But the Lorentz force depends on $\mathbf{v}$: $\mathbf{F}(t) \equiv \mathbf{F}(\mathbf{v}(t))$, and then the velocity update equation reads:

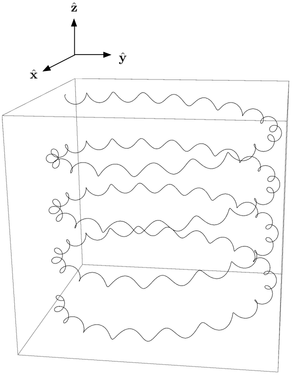
$$\mathbf{v}(t+\Delta t) = \mathbf{v}(t) + \frac{1}{2m}\left[\mathbf{F}(\mathbf{v}(t)) + \mathbf{F}(\mathbf{v}(t+\Delta t))\right]\Delta t$$

Because the Lorentz force is linear in $\mathbf{v}(t)$, one can (just barely) develop a consistent update to the Velocity Verlet algorithm that includes magnetic forcing [2].





Back to our problem, the numerical method gives the following trajectory:



(not what I expected)

Analysis

We'll avoid the equations of motion at all costs. Starting from the Lagrangian:

$$L = \frac{1}{2} m v^2 - qV + q\mathbf{v} \cdot \mathbf{A}.$$

Working in cylindrical coordinates, with $V = 0$ (for now)

$$L = \frac{1}{2} m \left(\dot{s}^2 + s^2 \dot{\phi}^2 + \dot{z}^2 \right) - \frac{\mu_0 q I}{2\pi} \log \left(\frac{s}{s_0} \right) \dot{z}.$$

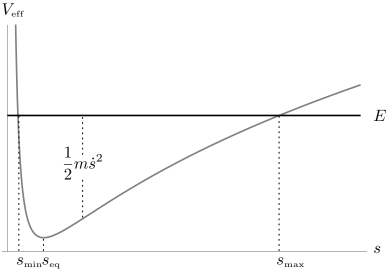
This L is independent of z and ϕ, so the associated canonical momenta are conserved - define $\alpha \equiv \frac{\mu_0 q I}{2\pi m}$:

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m s^2 \dot{\phi} = \ell$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} - m\alpha \log \left(\frac{s}{s_0} \right) = m\alpha \log \left(\frac{s_0}{s_1} \right)$$

Solve for $\dot{\phi}$ and $\dot{z}$, and use those in the constant kinetic energy expression (in cylindrical coordinates):

$$
\begin{aligned}
E &= \frac{1}{2} m \left(\dot{s}^2 + s^2 \dot{\phi}^2 + \dot{z}^2 \right) \\
&= \frac{1}{2} m \dot{s}^2 + \underbrace{\frac{\ell^2}{2ms^2} + \frac{1}{2} m \alpha^2 \log \left(\frac{s}{s_1} \right)^2}_{\equiv V_{\text{eff}}}
\end{aligned}
$$



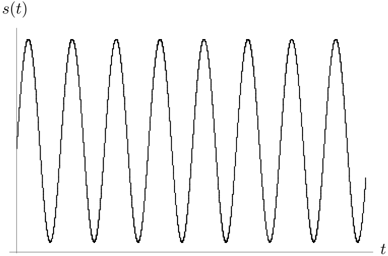
At the energy shown, the motion in s is oscillatory, with a period, T, defined by the effective potential. From

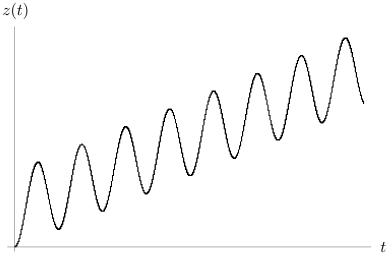
$$\dot{\phi} = \frac{\ell}{ms^2} \qquad \dot{z} = \alpha \log\left(\frac{s}{s_1}\right)$$

it is clear that both $\dot{\phi}$ and $\dot{z}$ will also be periodic with period T. The integral of $\dot{z}$ is of the form

$$z(t) = \sigma t + f(t)$$

where σ is a constant and $f(t)$ is the indefinite integral of the right-hand-side above, a function that is also periodic with period T. This most general type of motion is what we saw in the numerical solution - periodicity with “drift" in the z and ϕ coordinates of the particle.


s(t)
t

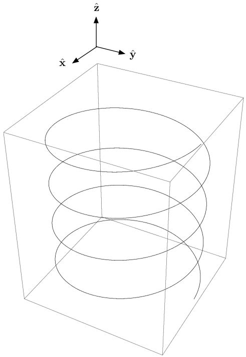

z(t)
t

No radial motion $\dot{s} = 0$

Our effective potential supports an s-equilibrium point, for which $\dot{s} = 0$ at s_{eq}. Then from the constants of motion, we have

$$\dot{\phi} = \frac{\ell}{m s_{eq}^2} \qquad \dot{z} = \alpha \log\left(\frac{s_{eq}}{s_1}\right),$$

also constants.

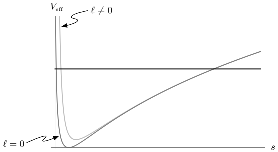


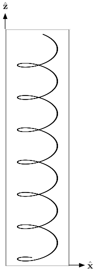
No angular momentum $\ell = 0$

We can also study the case of no angular momentum, $\ell = 0$. The effective potential becomes

$$V_{\text{eff}} = \frac{1}{2} m\alpha^2 \log\left(\frac{s}{s_1}\right)^2,$$

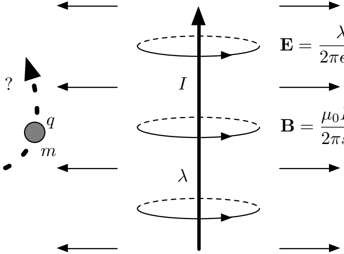
which still supports oscillatory behavior - now occurring in a plane.





The Problem - with Electric Field

Another opportunity for prediction:



$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}} \qquad V = -\frac{\lambda}{2\pi\epsilon_0}\log\left(\frac{s}{s_0}\right)$$

$$\mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}} \qquad \mathbf{A} = -\frac{\mu_0 I}{2\pi}\log\left(\frac{s}{s_0}\right)\hat{\mathbf{z}}$$

This time, the Lagrangian reads:

$$L = \frac{1}{2}m\left(\dot{s}^2 + s^2\dot{\phi}^2 + \dot{z}^2\right) + \frac{q\lambda}{2\pi\epsilon_0}\log\left(\frac{s}{s_0}\right) - \frac{\mu_0 q I}{2\pi}\log\left(\frac{s}{s_0}\right)\dot{z}$$

$$= \frac{1}{2}m\left(\dot{s}^2 + s^2\dot{\phi}^2 + \dot{z}^2\right) + m\alpha\left(\beta c - \dot{z}\right)\log\left(\frac{s}{s_0}\right),$$

with $\beta \equiv \frac{\lambda c}{I}$. The constants of the motion are unchanged, but we'll pick them differently here:

$$\frac{\partial L}{\partial \dot{\phi}} = ms^2\dot{\phi} = \ell$$

$$\frac{\partial L}{\partial \dot{z}} = m\dot{z} - m\alpha\log\left(\frac{s}{s_0}\right) = m\alpha\log\left(\frac{s_0}{s_1}\right) + m\beta c$$

Then the energy (kinetic plus potential, now) becomes

$$E = \frac{1}{2}mv^2 - m\alpha\beta c \log\left(\frac{ss_1}{s_1 s_0}\right)$$

$$= \frac{1}{2}m\dot{s}^2 + \frac{\ell^2}{2ms^2} + \frac{m}{2}\left(\beta c + \alpha \log\left(\frac{s}{s_1}\right)\right)^2 - m\alpha\beta c \log\left(\frac{s}{s_1}\right)$$

$$- m\alpha\beta c \log\left(\frac{s_1}{s_0}\right)$$

$$= \frac{1}{2}m\dot{s}^2 + \underbrace{\frac{\ell^2}{2ms^2} + \frac{m}{2}\alpha^2 \log\left(\frac{s}{s_1}\right)^2}_{\text{same } V_{\text{eff}} \text{ as before}} + E_0$$

There is an energy offset $E_0 = \frac{1}{2}m\beta^2c^2 - m\alpha\beta c \log(s_1/s_0)$, and a shift in p_z, but modulo those constants, no difference between the pure magnetic case and the magnetic + electrical case.

For the magnetic case:

$$m\dot{z} - m\alpha \log\left(\frac{s}{s_1}\right) = 0$$

gave

$$\dot{z} = \alpha \log\left(\frac{s}{s_1}\right).$$

In the magnetic $+$ electrical problem, we took

$$m\dot{z} - m\alpha \log\left(\frac{s}{s_1}\right) = m\beta c,$$

so that

$$\dot{z} = \beta c + \alpha \log\left(\frac{s}{s_1}\right),$$

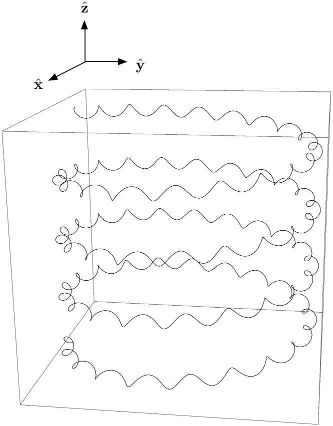
leading to a constant addition, βc, to the “drift" velocity in the z direction.

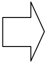
The moral of the story: If you had a solution to the magnetic problem $s_m(t)$, $\phi_m(t)$ and $z_m(t)$, then the solution to the magnetic $+$ electrical problem is

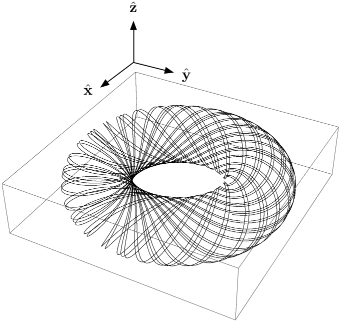
$$s_{em}(t) = s_m(t) \qquad \phi_{em}(t) = \phi_m(t) \qquad z_{em}(t) = z_m(t) + \beta c t,$$

with a shift in energy and (canonical) z-component of momentum that comes along for the ride.

By appropriate tuning of β, we can turn:







[1] Joel Franklin, David J. Griffiths & Nelia Mann, “Motion of a Charged Particle in the Static Fields of an Infinite Straight Wire," *Am. J. Phys.* **90**, 513 (2022).

[2] A. Chambliss & J. Franklin, “A magnetic velocity Verlet method," *Am. J. Phys.* **88**, 1075 (2020).