Lagrangian

In CM, we have a variety of options for expressing eqn.s of motion:

$$m\ddot{\vec{r}} = \vec{F} \Rightarrow m\ddot{x}(t) = F_x,\ m\ddot{y}(t) = F_y,\ m\ddot{z}(t) = F_z \quad (*)$$

these eqn.s look very different if they are written in, for example, spherical coordinates.

$x = r\sin\theta\cos\phi$ has:

$$\ddot{x} = \dot{r}\sin\theta\cos\phi + r\dot{\theta}\cos\theta\cos\phi - r\sin\theta\dot{\phi}\sin\phi$$

&

$$\ddot{x} = \dots$$

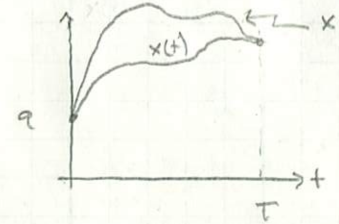
even if you could write these out easily – detangling the pieces from (*) would be "difficult."

The "action" framework makes it so that all eqn.s of motion "look" the same.

D=1 – an action $S[x(t)]$ takes a function $x(t)$ & returns a #:

$$S[x(t)] = \int_0^T \underbrace{L(x(t), \dot{x}(t))}_{\text{the integrand of the action}}\, dt$$

suppose we have some $x(t)$ that we like, & we want to change it slightly while keeping the end points fixed – i.e. leave $x(0) = a$, & $x(T) = b$ alone – maybe those are based on observations of some sort.



$\leftarrow x(t) + \epsilon f(t)$ w/ $f(0) = 0$, $f(T) = 0$

w/ $\epsilon \ll 1$.

? How does S respond to this change ?

$$S[x(t) + \epsilon f(t)] = \int_0^T L(x + \epsilon f, \dot{x} + \epsilon \dot{f})\, dt$$

$$\cong \underbrace{\int_0^T L(x, \dot{x})\, dt}_{= S[x(t)]} + \underbrace{\epsilon \int_0^T \left[\frac{\partial L}{\partial x} f + \frac{\partial L}{\partial \dot{x}} \dot{f}\right] dt}_{\equiv \delta S}$$

we can write the 2nd term of δS in terms of f itself using the b.c. for $f(t)$.

Note that $\displaystyle\int_0^T \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} f(t)\right) dt \overset{\text{fund. thm. of calc.}}{=} \left(\frac{\partial L}{\partial \dot{x}} f(t)\right)\Big|_{t=0}^{T} = 0$ $(f(0) = f(T) = 0)$

product rule $\parallel$ $\qquad\qquad\qquad \parallel$

$$\int_0^T f(t) \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) dt + \int_0^T \frac{\partial L}{\partial \dot{x}} \dot{f}\, dt = 0$$

so

$$\underbrace{\int_0^T \frac{\partial L}{\partial \dot{x}} \dot{f}\, dt}_{\text{our term in } \delta S} = -\int_0^T f(t) \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) dt$$

&

$$\delta S = \int_0^T \left[\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right] f(t)\, dt \quad (*)$$

We can define an "extremizing configuration" of S to be an $x(t)$ $\ni$

$$\delta S = 0 \quad \forall f(t)$$

a generalization of min. or max. to S.

(for a function $j(x)$, x is a min/max if

$j(x+\epsilon h) = j(x) + \epsilon h \frac{dj}{dx}$ has: $h\frac{dj}{dx} = 0 \ \forall h$...)

For δS in (*) to be zero $\forall f(t)$, we must have:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \qquad (\circ)$$

The details of this "eqn. of motion" depend on what you take L to be.

If $L = T - U = \frac{1}{2}m\dot{x}^2 - U$, a typical "Lagrangian" then the pieces of $(\circ)$ are:

$$\frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x} \qquad \frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \& \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$$

so $(\circ)$ reads:

$$-\frac{\partial U}{\partial x} - m\ddot{x} = 0 \implies m\ddot{x} = -\frac{dU}{dx} = F$$

which we recognize as Newton's 2^{nd} Law

If you have additional coord.s, like y & z, you can write:

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x,y,z)$$

& you have 3 eqn.s of motion now:

$$-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{\partial L}{\partial x} = 0, \quad -\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} + \frac{\partial L}{\partial y} = 0 \quad \& \quad -\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} + \frac{\partial L}{\partial z} = 0$$

recovering the 3 components of $\vec{F} = m\vec{a}$ in Cartesian coord.s

The real advantage is that the eqn.s of motion are coord. independent.

example: $L = T - U$ in cylindrical coord.s:

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(s,\phi,z)$$

$$x = s\cos\phi \implies \dot{x} = \dot{s}\cos\phi - s\dot{\phi}\sin\phi$$

$$y = s\sin\phi \implies \dot{y} = \dot{s}\sin\phi + s\dot{\phi}\cos\phi$$

$$\dot{x}^2 + \dot{y}^2 = \dot{s}^2 + s^2\dot{\phi}^2$$

&

$$L = \tfrac{1}{2}m(\dot{s}^2 + s^2\dot{\phi}^2 + \dot{z}^2) - U(s,\phi,z)$$

the eqn.s of motion are:

$$-\frac{d}{dt}\frac{\partial L}{\partial \dot{s}} + \frac{\partial L}{\partial s} = 0, \quad -\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} + \frac{\partial L}{\partial \phi} = 0, \quad -\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} + \frac{\partial L}{\partial z} = 0$$

Noether's Theorem

Suppose we have a Lagrangian like

$$L = \tfrac{1}{2} m(\dot{s}^2 + s^2\dot{\phi}^2 + \dot{z}^2) - U(s) \qquad (*)$$

i.e. an L that is independent of ϕ & z, then

$L(z+a) = L(z)$, the L is unchanged by $z \to z+a$ so $z \to z+a$ is a "symmetry" of L.
translation

From the eqn. of motion,

$$-\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} + \cancel{\frac{\partial L}{\partial z}} = 0 \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{z}} = 0$$

$\frac{\partial L}{\partial \dot{z}} = m\dot{z}$ is a constant of the motion

Noether's theorem: symmetry $\Rightarrow$ conservation

Similarly, for $(*)$, we have L unchanged by $\phi \to \phi + \varphi$ a rotation about the z-axis. That symmetry leads to a conserved quantity:

$$-\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} + \cancel{\frac{\partial L}{\partial \phi}} = 0 \Rightarrow \frac{\partial L}{\partial \dot{\phi}} = ms^2\dot{\phi} = L_z$$

z - angular momentum

E & M

$L = T - U$ for potential energy U - easy to introduce an electrostatic potential: $U = qV$
pot'l.

but how to incorporate $\vec{B}$?

We know that $A^\mu \doteq \begin{pmatrix} V/c \\ \vec{A} \end{pmatrix}$ & $\dot{x}_\mu \doteq \begin{pmatrix} -c \\ \vec{v} \end{pmatrix}$

so

$$q \sum_\mu A^\mu \dot{x}_\mu = -qV + q\vec{v}\cdot\vec{A}$$

this quantity is what appears in L. Presumably, $q\vec{v}\cdot\vec{A}$ should come along for the ride ...

$$L = \tfrac{1}{2} mv^2 - qV + q\vec{v}\cdot\vec{A}$$

the eqn.s of motion here lead directly to

$$m\ddot{\vec{r}} = q\vec{E} + q\vec{v}\times\vec{B}$$
$$= q\left(-\nabla V - \frac{\partial \vec{A}}{\partial t}\right) + q\vec{v}\times(\nabla\times\vec{A})$$