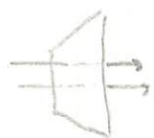
Rutherford Scattering

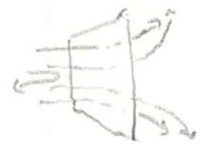
Thomson's "plum pudding model" of the atom — positive charges w/ e^- "floating" in it

$\rho = \frac{Q}{4/3\pi R^3}$

If you shoot positive charges at a sheet of these atoms, they should go right through

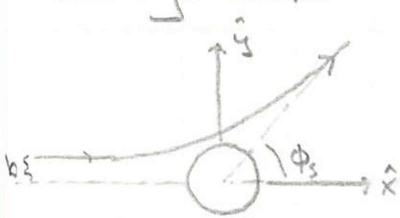


but instead



evidence for charge localization w/in the atom — a "nucleus" where the protons live.

Scattering Setup



Given the "impact parameter," b, find the "scattering angle" Φ_s.

In $D=3$:

$$L = \tfrac{1}{2}mv^2 - U = \tfrac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right] - U$$

If U is a function of r (spherical symmetry), angular momentum conservation $\Rightarrow$ motion occurs in a plane — take it to be the x-y plane:

$$\theta = \pi/2 \quad , \quad \dot{\theta} = 0$$

we have eqn. of motion, for ϕ:

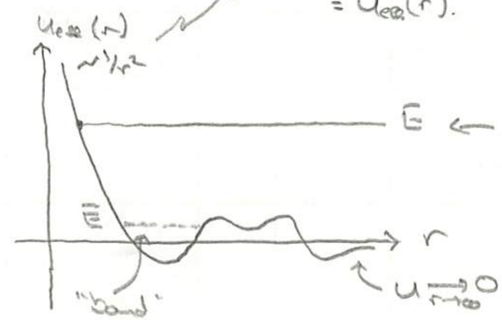
$$-\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} + \cancel{\frac{\partial L}{\partial \phi}} = 0 \Rightarrow \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} = \text{const}$$

what should we call it? $mr^2\dot{\phi} = L_z \Rightarrow \dot{\phi} = \frac{L_z}{mr^2}$.

The total (conserved) energy is:

$$E = \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\phi}^2 + U(r)$$

$$= \tfrac{1}{2}m\dot{r}^2 + \underbrace{\frac{L_z^2}{2mr^2} + U(r)}_{\equiv U_{eff}(r)}$$



$E - U_{eff}(r) = \tfrac{1}{2}m\dot{r}^2$

E ← scattering story of motion (radial): particle comes in, stops, heads back out

Motivated by the $\frac{L_z^2}{2mr^2}$ term, & the requirement that $U \xrightarrow[r\to\infty]{} 0$ (& target Coulomb potential),

let $r = 1/\rho$, w/ $\dot{r} = -\frac{1}{\rho^2}\dot{\rho}$, &

$$E = \tfrac{1}{2}m\left(\frac{\dot{\rho}}{\rho^2}\right)^2 + \frac{L_z^2}{2m}\rho^2 + U(\rho) \leftarrow \text{put in } r(\rho)$$

The target figure does not require the temporal evolution — we could use ϕ as the parameter to get a geometric picture of the scattering trajectory.

For $\rho(\phi(t))$, we have:

$$\frac{d\rho}{dt} = \frac{d\rho}{d\phi}\frac{d\phi}{dt} = \rho'(\phi)\,\dot{\phi} = \rho'(\phi)\frac{L_z}{m}\rho^2$$

& our energy expression is:

$$E = \frac{L_z^2}{2m}\left[\rho'(\phi)^2 + \rho(\phi)^2\right] + U \qquad (+)$$

Take the ϕ-derivative of both sides to get:

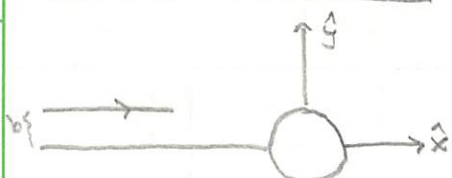
$$0 = \frac{L_z^2}{m}\left[\rho''(\phi) + \rho(\phi)\right]\rho'(\phi) + \frac{dU}{d\rho}\cdot\rho'(\phi)$$

or

$$\rho''(\phi) = -\rho(\phi) - \frac{m}{L_z^2}\frac{dU}{d\rho} \qquad (\circ)$$

as the "eqn. of motion". Since it's 2^{nd} order, we'll need a pair of initial conditions.

<u>Initial</u> <u>Conditions</u>



At "$t=-\infty$", we have:

$x(-\infty) = -\infty$, $y(-\infty) = b$

$\dot{x}(-\infty) = v_0$ $\dot{y}(-\infty) = 0$

w/ $E = \frac{1}{2}mv_0^2$ (always) &

$$L_z = (\vec{r}\times\vec{p})\big|_z \text{ "}= m[-\infty\hat{x} + b\hat{y}][v_0\hat{x}]\text{"} = -mv_0 b\,\hat{z}$$

or $L_z^2 = mb^2(mv_0^2) = 2Emb^2$

? in terms of ϕ, $t \to -\infty \Rightarrow \phi \to$?

w/ $\rho(\pi) = 0$

& $E = \frac{L_z^2}{2m}\rho'(\pi)^2$ (from (+))

$$\Downarrow$$

$$\rho'(\pi) = -\sqrt{\frac{2mE}{L_z^2}} = \pm\frac{1}{b}.$$

=

<u>Coulomb</u> <u>Potential</u>

For $U = \frac{kq^2}{r}$ $\left(k = \frac{1}{4\pi\epsilon_0}\right)$

$= kq^2\rho$

($\circ$) reads: $\rho'' = -\rho - \frac{m}{2Emb^2}\cdot kq^2 = -\rho - \frac{kq^2}{2Eb^2}$.

$$\rho(\phi) = A\cos\phi + B\sin\phi - \frac{kq^2}{2Eb^2}.$$

w/ $\rho(\pi) = -A - \frac{kq^2}{2Eb^2} = 0$

& $\rho'(\pi) = -B = \frac{-1}{b}$.

so

$$\rho(\phi) = -\frac{kq^2}{2Eb^2}\left(1 + \cos\phi\right) + \frac{1}{b}\sin\phi$$

At the scattering angle, ϕ_s, we have

$$\rho(\phi_s) = 0$$

we can use this defining feature to express $b(\Phi_s)$:

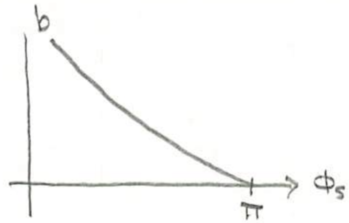
$$p(\Phi_s) = -\frac{kq^2}{2Eb^2}(1+\cos\Phi_s) + \frac{1}{b}\sin\Phi_s = 0$$

gives

$$b = \frac{kq^2}{2E}\underbrace{\frac{(1+\cos\Phi_s)}{\sin\Phi_s}}_{=\cot(\Phi_s/2)}$$

$$= \frac{q^2}{8\pi\epsilon_0 E}\cot(\Phi_s/2)$$

which can be inverted if desired.

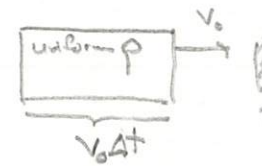


- $\Phi_s = 0$ is the undeflected path, happens as $b \longrightarrow \infty$.
- $b = 0$ is the "head on" collision, w/ $\Phi_s = \pi$.

- For a given Φ_s, increasing the energy of the particle decreases the impact parameter.

Scattering Cross Section

Now you send in a beam of particles w/ "luminosity" $\mathcal{L}$ (# of particles/area/time), w/ $\rho \equiv$ #/vol. particles traveling at speed v_0



# of particles going through the disk in time Δt is

$$N = \rho \cdot A \cdot (v_0 \Delta t) = \underbrace{\rho v_0}_{=\mathcal{L}} (A \Delta t)$$

For a portion of the beam w/ area $d\sigma$,

detector area $h^2 \underbrace{\sin\theta\, d\theta\, d\phi}_{d\Omega}$

we have: $\mathcal{L}\, d\sigma = \mathcal{L}' h^2 d\Omega \Rightarrow d\sigma = \underbrace{\frac{\mathcal{L}' h^2}{\mathcal{L}}}_{\text{"differential scattering cross section"}} d\Omega$