Combined Stress Tensors

We had:

$$T^{\mu\nu}_{EM} = \frac{1}{\mu_0}\left[F^{\mu\sigma}F^{\nu}{}_{\sigma} - \tfrac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right]$$

for the electromagnetic field, 6

$$T_p^{\mu\nu} = \frac{m v^\mu v^\nu}{\sqrt{1-v^2/c^2}}\,\delta^3(\vec{r}-\vec{r}(t))$$

When we combine the two:

$$T^{\mu\nu} = T^{\mu\nu}_{EM} + T_p^{\mu\nu}$$

we have energy/momentum conservation.

The conservation of momentum piece looks like:

$\partial_\mu T^{\mu j} = 0$ for $j = 1, 2, 3$.

$$\frac{1}{c}\frac{\partial T^{0j}}{\partial t} + \frac{\partial T^{ij}}{\partial x^i} = 0 \quad \text{applied to:}$$

we have:

$$\frac{d}{dt}\int_\Omega \frac{T^{0j}}{c}\,d\tau = -\oint_{\partial\Omega} T^{ij}\,da_i$$

$$\frac{d}{dt}\int_\Omega \underbrace{\frac{1}{c^2}S^j}_{=\epsilon_0(\vec{E}\times\vec{B})^j}\,d\tau + \frac{d}{dt}\frac{m v^j}{\sqrt{1-v^2/c^2}} = -\oint_{\partial\Omega} T^{ij}_{EM}\,da_i$$

? where did the "surface term" for the particles go?

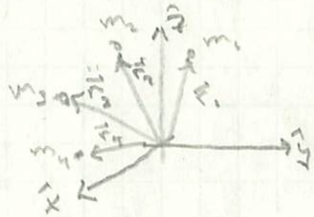
This looks like Newton's 2^{nd} Law (relativistic form: $\vec{p} = \frac{m\vec{v}}{\sqrt{1-v^2/c^2}}$)

$$\frac{d\vec{p}}{dt} = -\frac{d}{dt}\int_\Omega \epsilon_0(\vec{E}\times\vec{B})\,d\tau - \oint_{\partial\Omega} \overleftrightarrow{T}_{EM}\cdot d\vec{a}$$

so that $-\overleftrightarrow{T}_{EM}$ is the force per unit area we developed earlier (8.17) in the textbook.

$$\overleftrightarrow{T}_{Maxwell} = -\overleftrightarrow{T}_{EM}$$

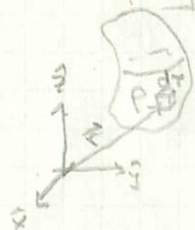
Center of Mass



For a set of particles w/ masses $\{m_i\}_{i=1}^n$ at locations $\{\vec{r}_i\}_{i=1}^n$:

$$\vec{R}_{CM} = \frac{1}{M}\sum_{i=1}^n m_i\vec{r}_i \quad \text{w/ } M = \sum_{i=1}^n m_i$$

For a continuous distribution of mass described by mass density $\rho(\vec{r})$



$$dm = \rho(\vec{r})\,d\tau, \quad M = \int_\Omega \rho(\vec{r})\,d\tau$$

$$\vec{R}_{cm} = \frac{1}{M}\int_\Omega \vec{r}\,dm = \frac{1}{M}\int_\Omega \rho(\vec{r})\,\vec{r}\,d\tau.$$

The advantage of $\vec{R}_{cm}$ is: $M\frac{d\vec{R}_{cm}}{dt} = \sum_{i=1}^n m_i\frac{d\vec{r}_i}{dt} \equiv \vec{P}$, the total momentum, so if

$\frac{d\vec{R}_{cm}}{dt} = 0$ then $\vec{P} = 0$ – if the center of mass is not moving, there is no net momentum. (same for continuous version).

Center of Energy

When we combine particles w/ fields, we can't use the center of mass (fields don't have mass) but we can define the center of energy, & it will play a similar role.

$$T^{\mu\nu} = T^{\mu\nu}_{EM} + T^{\mu\nu}_{p} \quad \text{has} \quad T^{00} = u_{EM} + u_p$$

$$\text{w/ } u_{EM} = \tfrac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0} B^2 \quad \& \quad u_p = \frac{mc^2}{\sqrt{1-v^2/c^2}}\,\delta^3(\vec r - \vec r(t)).$$

& sum.

$$T^{0i} = c\,(g^i_{EM} + p^i_p) \quad \text{the combined momentum densities.}$$

With:

$$\frac{1}{c}\frac{d}{dt}\underbrace{\int_\Omega T^{00}\, d\tau}_{=E} = -\oint_{\partial\Omega} T^{0i}\, da_i$$

$$\frac{d}{dt}\underbrace{\int_\Omega \frac{T^{0j}}{c}\, d\tau}_{p^j} = -\oint_{\partial\Omega} T^{ij}\, da_i$$

Define a "closed system" to be one which

$$T^{\mu\nu}\big|_{\partial\Omega} = 0 \quad \text{(stress tensor vanishes on boundary).}$$

example – the universe ...

then $\frac{dE}{dt} = 0$ & $\frac{d\vec p}{dt} = 0$

total energy & momentum are conserved.

Define the "center of energy" by analogy w/ center of mass:

$$\vec R_E = \frac{1}{E}\int_\Omega T^{00}\,\vec r\, d\tau \quad \text{w/ } E = \int_\Omega T^{00}\, d\tau$$

then for a closed system,

$$\frac{d\vec R_E}{dt} = \frac{1}{E}\int_\Omega \frac{\partial T^{0i}}{\partial t}\,\vec r\, d\tau \qquad \left(\frac{dE}{dt} = 0 \text{ for closed system}\right)$$

$$\to \frac{1}{c}T^{00} + \frac{\partial T^{0i}}{\partial x^i} = 0$$

$$\frac{d\vec R_E}{dt} = -\frac{c}{E}\int_\Omega \frac{\partial T^{0i}}{\partial x^i}\,\vec r\, d\tau \qquad (*)$$

We'll need the following integration-by-parts identity:

$$\int_\Omega \frac{\partial}{\partial x^i}\left(T^{0i} r^j\right) d\tau = \oint_{\partial\Omega} T^{0i} r^j\, da_i \overset{\text{closed system}}{=} 0$$

" product rule "

$$\int_\Omega \frac{\partial T^{0i}}{\partial x^i}\, r^j\, d\tau + \int_\Omega T^{0i}\underbrace{\frac{\partial r^j}{\partial x^i}}_{=\delta^j_i} d\tau = 0$$

$$\text{so} \quad \int_\Omega \frac{\partial T^{0i}}{\partial x^i}\, r^j\, d\tau = -\int_\Omega T^{0j}\, d\tau$$

apply this to (*) written w/ indices:

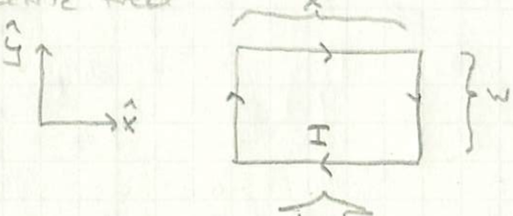
$$\frac{dR^j_E}{dt} = -\frac{c}{E}\int_\Omega \frac{\partial T^{0i}}{\partial x^i}\, r^j\, d\tau = \frac{c}{E}\int_\Omega T^{0j}\, d\tau = \frac{c^2}{E}p^j \qquad (+)$$

Hidden Momentum

The physical lesson of (*) is that if the center of energy is at rest, the total momentum is zero; & vice versa.

$$\frac{d\vec{R}_E}{dt} = 0 \iff \vec{p} = 0.$$

Consider the following configuration: a loop of wire carrying steady current I has a dipole moment $\vec{m} = I\vec{a}$. It is placed in a uniform electric field



$$\vec{m} = Iw\ell(-\hat{z})$$

$$\vec{E} = E\hat{y}$$

The dipole generates a magnetic field, & together w/ the electric field, there is momentum density:

$$\vec{g} = \epsilon_0 \vec{E}\times\vec{B}$$

& total momentum

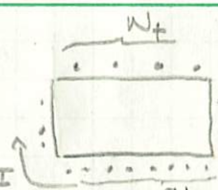
$$\vec{p}_{EM} = -\frac{1}{c^2}(\vec{m}\times\vec{E})$$

This is the field momentum – it is not zero...

The center of energy is at rest, so the particle momentum must be

$$\vec{p}_p = \frac{1}{c^2}(\vec{m}\times\vec{E}) \ldots$$

looking inside the wire:



charges speed up going up the left side, & slow down coming down the right (due to the electric force)

There are N_+ charges going at speed u_+ on the top segment, N_b charges going u_b along the bottom – each charge is q

$$I_+ = \lambda_+ v_+ = q\frac{N_+}{\ell}\cdot u_+ \quad \& \quad I_b = \lambda_b v_b = q\frac{N_b}{\ell}u_b$$

the current is steady, so

$$I_+ = I_b = I \Rightarrow N_+u_+ = N_bu_b = \frac{I\ell}{q}$$

then the non relativistic momentum is: $\vec{p}_p = (mN_+u_+ - mN_bu_b)\hat{x} = 0$ (for particles of mass m).

But the center of energy theorem is for relativistic momentum &

$$\vec{p}_p = \left[\frac{mN_+u_+}{\sqrt{1-u_+^2/c^2}} - \frac{mN_bu_b}{\sqrt{1-u_b^2/c^2}}\right]\hat{x} = m(N_+u_+\gamma_+ - N_bu_b\gamma_b)\hat{x}$$

$$= \frac{mI\ell}{q}(\gamma_+ - \gamma_b)\hat{x}$$

the change in energy of the particle going from the bottom to the top is:

$$\Delta E = mc^2\gamma_+ - mc^2\gamma_b = W = qEw \Rightarrow \gamma_+ - \gamma_b = \frac{qEw}{mc^2}$$

so $\vec{p}_p = \frac{mI\ell}{q}\frac{qEw}{mc^2}\hat{x} = \frac{I\ell w}{c^2}E\hat{x} = \frac{1}{c^2}(\vec{m}\times\vec{E})$ ✓