Stress Tensor for Vector Fields

Our vector field $\mathcal{L} = \frac{1}{4\mu_0} F^{\rho\beta} F_{\rho\beta}$, a scalar that depends on the field strength tensor:

$$F_{\rho\beta} \equiv \partial_\rho A_\beta - \partial_\beta A_\rho$$

w/ field eqn: $\partial^\mu \partial_\nu A^\nu - \partial^\nu \partial_\nu A^\mu = 0$

This eqn. is unchanged by $x^\mu \to x^\mu + \underbrace{d^\mu}_{\text{constants}}$ so the basic stress tensor story from scalar fields holds here.

For scalars: $\phi \to \phi + \theta$, $\mathcal{L} \to \mathcal{L} + \partial_\mu K^\mu$ gave

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \theta - K^\mu$$

For vectors: If $A_\sigma \to A_\sigma + \psi_\sigma$ leaves the field eqn. unchanged,

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\sigma)} \psi_\sigma - K^\mu$$

sum over σ

What ψ_σ is induced by $x^\mu \to x^\mu + d^\mu$?

$$A_\sigma(x^\mu + d^\mu) \approx A_\sigma(x^\mu) + \underbrace{(\partial_\alpha A_\sigma) d^\alpha}_{\equiv \psi_\sigma}$$

+ $\mathcal{L} \to \mathcal{L} + \partial_\mu \mathcal{L} d^\mu \Rightarrow K^\mu = \mathcal{L} d^\mu$ as before

So $J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\sigma)} \partial_\alpha A_\sigma d^\alpha - \mathcal{L} \underbrace{d^\mu}_{= \delta^\mu_\alpha d^\alpha}$

$$= d^\alpha \underbrace{\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\sigma)} \partial_\alpha A_\sigma - \delta^\mu_\alpha \mathcal{L} \right]}_{\equiv T^\mu{}_\alpha}$$

For our $\mathcal{L}$, we need:

$$\frac{\partial}{\partial(\partial_\mu A_\sigma)} \left[\frac{1}{4\mu_0} F^{\rho\beta} F_{\rho\beta} \right] = \frac{1}{2\mu_0} F^{\rho\beta} \frac{\partial F_{\rho\beta}}{\partial(\partial_\mu A_\sigma)}$$

+ $$\frac{\partial F_{\rho\beta}}{\partial(\partial_\mu A_\sigma)} = \frac{\partial}{\partial(\partial_\mu A_\sigma)} \left[\partial_\rho A_\beta - \partial_\beta A_\rho \right]$$

$$= \delta^\mu_\rho \delta^\sigma_\beta - \delta^\mu_\beta \delta^\sigma_\rho$$

so $$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\sigma)} = \frac{1}{2\mu_0} \left[F^{\mu\sigma} - \underbrace{F^{\sigma\mu}}_{= -F^{\mu\sigma}} \right] = \frac{1}{\mu_0} F^{\mu\sigma}$$

$$T^\mu{}_\alpha = \frac{1}{\mu_0} F^{\mu\sigma} \partial_\alpha A_\sigma - \frac{1}{4\mu_0} \delta^\mu_\alpha F^{\rho\beta} F_{\rho\beta}$$

+

$$T^{\mu\alpha} = \frac{1}{\mu_0} F^{\mu\sigma} \partial^\alpha A_\sigma - \frac{1}{4\mu_0} \eta^{\mu\alpha} F^{\rho\beta} F_{\rho\beta}$$

To $T^{\mu\alpha}$ we can add any term w/ zero div. to the

$f^{\sigma\mu\alpha}$ w/ $f^{\mu\sigma\alpha} = -f^{\sigma\mu\alpha}$, then

$\tilde{T}^{\mu\alpha} = T^{\mu\alpha} + \partial_\sigma f^{\sigma\mu\alpha}$

has:

$$\partial_\mu \tilde{T}^{\mu\alpha} = \underbrace{\partial_\mu T^{\mu\alpha}}_{=0 \text{ automatically}} + \overbrace{\partial_\mu \partial_\sigma f^{\sigma\mu\alpha}}^{=0 \text{ by symmetry}}$$

Let $f^{\sigma\mu\alpha} = \frac{1}{\mu_0} F^{\sigma\mu} A^\alpha$

Then

$$\tilde{T}^{\mu\alpha} = \frac{1}{\mu_0}\left[F^{\mu\sigma}\partial^\alpha A_\sigma - F^{\mu\sigma}\partial_\sigma A^\alpha - \tfrac{1}{4}\eta^{\mu\alpha}F^{\rho\beta}F_{\rho\beta}\right]$$

$$= \frac{1}{\mu_0}\left[F^{\mu\sigma}F^{\alpha}{}_{\sigma} - \tfrac{1}{4}\eta^{\mu\alpha}F^{\rho\beta}F_{\rho\beta}\right]$$

Combined Particles & Fields.

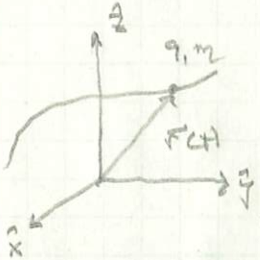
We've seen that energy & momentum conservation requires both particles & fields for a complete accounting.

We can tell that story using the stress tensor

$$T^{\mu\nu}_{EM} = \frac{1}{\mu_0}\left[F^{\mu\sigma}F^{\nu}{}_{\sigma} - \tfrac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right]$$

But we need to augment w/ a particle stress tensor ...

For a particle travelling along $\vec{r}(t)$, the energy density is:



$$u = \frac{mc^2}{\sqrt{1-v^2/c^2}}\,\delta^3(\vec{r}-\vec{r}(t))$$

& the momentum density is:

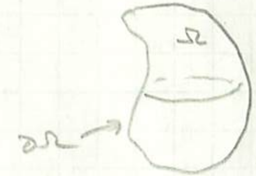
$$\vec{g} = \frac{m\vec{v}}{\sqrt{1-v^2/c^2}}\,\delta^3(\vec{r}-\vec{r}(t))$$

& these are part of the larger:

$$T_p^{\mu\nu} = \frac{m v^\mu v^\nu}{\sqrt{1-v^2/c^2}}\,\delta^3(\vec{r}-\vec{r}(t))$$

w/ $v^\mu \doteq \begin{pmatrix} c \\ \vec{v}\end{pmatrix}$

conservation for a volume Ω looks like:



$$\partial_\mu T^{\mu\nu}_{EM} = 0 \Rightarrow \frac{d}{dt}\int_\Omega \frac{T^{00}}{c}\,d\tau = -\oint_{\partial\Omega} T^{0j}\,da_j$$

$$\& \quad \frac{d}{dt}\int_\Omega \frac{T^{i0}}{c}\,d\tau = -\oint_{\partial\Omega} T^{ij}\,da_j$$

this second one looks like

$$\frac{d}{dt}\,\frac{m\vec{v}}{\sqrt{1-v^2/c^2}} = 0 \qquad (\delta\text{'s vanish on } \partial\Omega;\ \text{particle is } \underline{\text{in}}\ \Omega).$$

That's Newton's 2^{nd} law w/ no forces.

Let $T^{\mu\nu} = T_p^{\mu\nu} + T^{\mu\nu}_{EM}$

Conservation holds for the entire system, so

$$\frac{d}{dt}\int_\Omega \frac{T^{i0}}{c}\,d\tau = -\oint_{\partial\Omega} T^{ij}\,da_j \Rightarrow \frac{d\vec{p}}{dt} = -\oint_{\partial\Omega} T^{ij}_{EM}\,da_j$$

w/ $\vec{p}$ the total momentum (particle + field) in Ω

Now $-T^{ij}_{EM}$ looks like the EM force/area acting on the particle in Ω.

$T^{ij}_{EM} = -T^{ij}_{Maxwell}$ ⟵ the Maxwell stress tensor we've seen before.