Stress Tensor for Vector Fields

Our vector field $\mathcal{L} = \frac{1}{4\mu_0} F^{\rho\beta} F_{\rho\beta}$, a scalar that depends on the field strength tensor:

$$F_{\rho\beta} \equiv \partial_\rho A_\beta - \partial_\beta A_\rho$$

w/ field eqn.: $\partial^\mu \partial_\nu A^\nu - \partial^\nu \partial_\nu A^\mu = 0$

This eqn. is unchanged by $x^\mu \to x^\mu + \underset{\text{constants}}{d^\mu}$ so the basic stress tensor story from scalar fields holds here.

For scalars: $\phi \to \phi + \theta$, $\mathcal{L} \to \mathcal{L} + \partial_\mu K^\mu$ gave

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \theta - K^\mu$$

For vectors: If $A_\sigma \to A_\sigma + \Psi_\sigma$ leaves the field eqn. unchanged,

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\sigma)} \cdot \Psi_\sigma - K^\mu$$

sum over σ

What Ψ_σ is induced by $x^\mu \to x^\mu + d^\mu$?

$$A_\sigma(x^\mu + d^\mu) \simeq A_\sigma(x^\mu) + \underbrace{(\partial_\alpha A_\sigma) d^\alpha}_{\equiv \Psi_\sigma}$$

$\Rightarrow \mathcal{L} \to \mathcal{L} + \partial_\mu \mathcal{L} d^\mu \Rightarrow K^\mu = \mathcal{L} d^\mu$ as before.

So $J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\sigma)} \partial_\alpha A_\sigma d^\alpha - \mathcal{L} \underset{= \delta^\mu_\alpha d^\alpha}{d^\mu}$

$$= d^\alpha \underbrace{\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\sigma)} \partial_\alpha A_\sigma - \delta^\mu_\alpha \mathcal{L} \right]}_{\equiv T^\mu{}_\alpha}$$

For our $\mathcal{L}$, we need:

$$\frac{\partial}{\partial(\partial_\mu A_\sigma)} \left[\frac{1}{4\mu_0} F^{\rho\beta} F_{\rho\beta} \right] = \frac{1}{2\mu_0} F^{\rho\beta} \frac{\partial F_{\rho\beta}}{\partial(\partial_\mu A_\sigma)}$$

$$\& \quad \frac{\partial F_{\rho\beta}}{\partial(\partial_\mu A_\sigma)} = \frac{\partial}{\partial(\partial_\mu A_\sigma)} \left[\partial_\rho A_\beta - \partial_\beta A_\rho \right]$$

$$= \delta^\mu_\rho \delta^\sigma_\beta - \delta^\mu_\beta \delta^\sigma_\rho$$

so $\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\sigma)} = \frac{1}{2\mu_0} \left[F^{\mu\sigma} - \underset{= -F^{\mu\sigma}}{F^{\sigma\mu}} \right] = \frac{1}{\mu_0} F^{\mu\sigma}$

$$T^\mu{}_\alpha = \frac{1}{\mu_0} F^{\mu\sigma} \partial_\alpha A_\sigma - \frac{1}{4\mu_0} \delta^\mu_\alpha F^{\rho\beta} F_{\rho\beta}$$

$\&$

$$T^{\mu\alpha} = \frac{1}{\mu_0} F^{\mu\sigma} \partial^\alpha A_\sigma - \frac{1}{4\mu_0} \eta^{\mu\alpha} F^{\rho\beta} F_{\rho\beta}$$

To $T^{\mu\alpha}$ we can add any term w/ zero div.
take

$f^{\sigma\mu\alpha}$ w/ $f^{\mu\sigma\alpha} = -f^{\sigma\mu\alpha}$, then

$$\tilde{T}^{\mu\alpha} = T^{\mu\alpha} + \partial_\sigma f^{\sigma\mu\alpha}$$

has:

$$\partial_\mu \tilde{T}^{\mu\alpha} = \underbrace{\partial_\mu T^{\mu\alpha}}_{=0 \text{ automatically}} + \overbrace{\partial_\mu \partial_\sigma f^{\sigma\mu\alpha}}^{=0 \text{ by symmetry}}$$

Let $f^{\sigma\mu\alpha} = \frac{1}{\mu_0} F^{\sigma\mu} A^\alpha$

Then

$$\tilde{T}^{\mu\alpha} = \frac{1}{\mu_0}\left[F^{\mu\sigma} \partial^\alpha A_\sigma - F^{\mu\sigma} \partial_\sigma A^\alpha - \tfrac{1}{4} \eta^{\mu\alpha} F^{\rho\beta} F_{\rho\beta} \right]$$

$$= \frac{1}{\mu_0}\left[F^{\mu\sigma} F^\alpha{}_\sigma - \tfrac{1}{4} \eta^{\mu\alpha} F^{\rho\beta} F_{\rho\beta} \right]$$