Infinitesimal Transformations

Both rotations & boosts can be written in terms of infinitesimal parameters. The inf. transformation takes the form

$$\bar{x}^\mu = x^\mu + \epsilon M^\mu{}_\nu x^\nu$$

inf. param $\ll 1$ — ϵ; matrix w/ ± 1 or 0 as entries — $M^\mu{}_\nu$

How does ϕ respond?

$$\phi(x^\mu + \epsilon M^\mu{}_\nu x^\nu) \approx \phi(x^\mu) + \underbrace{\epsilon M^\mu{}_\nu x^\nu \partial_\mu \phi}_{\equiv \Theta}$$

& sim., $K^\mu = \epsilon M^\mu{}_\nu x^\nu \mathcal{L}$.

The conserved current is:

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Theta - K^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \epsilon M^\rho{}_\nu x^\nu \partial_\rho \phi - \epsilon M^\mu{}_\nu x^\nu \mathcal{L}$$

$$= \epsilon M^\rho{}_\nu x^\nu \underbrace{\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\rho \phi - \delta^\mu_\rho \mathcal{L}\right]}_{\equiv T^\mu{}_\rho}$$

$\equiv T^\mu{}_\rho$ the stress-tensor from last time.

so

$$J^\mu = \epsilon M^\rho{}_\nu x^\nu T^\mu{}_\rho = \epsilon M_{\rho\nu}[x^\nu T^{\mu\rho}]$$

For rotations, $M_{\rho\nu} = -M_{\nu\rho}$ ("matrix" is antisymmetric) & we have:

$$J^\mu = \tfrac{1}{2} \epsilon M_{\rho\nu}[x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu}]$$

only antisym in $\rho \leftrightarrow \nu$ part of $x^\nu T^{\mu\rho}$ survives the sum.

Vector Field Action

$$S[A_\mu] = \int_\Omega \mathcal{L}(A, \partial A)\, d^4x$$

assume the same action structure – but now the $\mathcal{L}$ depends on A_μ & $\partial_\mu A_\nu$.

The S-extremizing "Euler-Lagrange" eqn.s are just 4 copies of

$$-\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) + \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

For A_α $(\alpha = 0, 1, 2, 3)$:

$$-\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\alpha)}\right) + \frac{\partial \mathcal{L}}{\partial A_\alpha} = 0$$

That's the set of four $(\alpha = 0 \to 3)$ field eqn.s. How about $\mathcal{L}$ – how should we construct the action?

Model Building Guidelines

Our target field eqn.s have the following desirable properties – in order of importance.

1. Field eqn.s should be Lorentz-covariant (same form in L & $\bar{L}$) $\Rightarrow \mathcal{L}$ should be a scalar (no "open" indices).

3. Field eqn.s should not make reference to lower-rank fields $\Rightarrow \mathcal{L}$ should not depend on lower-rank contributions.

A vector, like A^μ is a "1st rank" (1-index) field.

this rule suggests that we do not want $\mathcal{L}$ to depend on 0-rank (no indices), scalar, fields.

Makes sense – we already have a theory of scalar fields, keep 'em separate.

2. field eqn.s should be 2nd order in space & time derivatives $\Rightarrow$ $\mathcal{L}$ depends on at most 1st deriv.s of the field.

the motivation here is one of familiarity – all of the field eqn.s we know (& love) are 2nd order: $\Box V = -\rho/\epsilon_0$.

4. field eqn.s should be linear in the fields, so $\mathcal{L}$ should be at most quadratic in the fields & their derivatives.

this one is for simplicity – linear field eqn.s support superposition.

It's often the case that field eqn.s "become" nonlinear "later on" (when we smash them together), so our target starting point will be linear versions.

Let's think about the available terms for the field eqn:

$$\partial^\mu \partial_\nu A^\nu \ , \quad \partial_\nu \partial^\nu A^\mu \ , \quad A^\mu$$

The most general field eqn. we could make is:

$$\alpha \partial^\mu \partial_\nu A^\nu + \beta \partial_\nu \partial^\nu A^\mu + \sigma A^\mu = 0 \qquad (*)$$

for constants $\{\alpha, \beta, \sigma\}$.

$(*)$ is linear & at-most 2nd derivatives appear, it holds in both L & $\bar{L}$.

How about "requirement" 3. – does $(*)$ constrain scalar fields?

How might a scalar field be disguised as a vector field?

$$A^\mu = \partial^\mu \phi$$

idea – put $A^\mu = \partial^\mu \phi$ in to $(*)$ & require that the result vanish, "0=0".

$$\alpha \underbrace{\partial^\mu \partial_\nu (\partial^\nu \phi)}_{= \partial_\nu \partial^\nu \partial^\mu \phi} + \beta \partial_\nu \partial^\nu \partial^\mu \phi + \sigma \partial^\mu \phi = 0$$

same term

so

$$(\alpha + \beta) \partial_\nu \partial^\nu \partial^\mu \phi + \sigma \partial^\mu \phi = 0$$

If we want the LHS to vanish, we need $\sigma = 0$, $\beta = -\alpha$ & our field eqn. becomes:

$$\alpha \left[\partial^\mu \partial_\nu A^\nu - \partial_\nu \partial^\nu A^\mu \right] = 0 \qquad (+)$$

This is our target field eqn. – we can use it to construct the $\mathcal{L}$ – there is no A^μ term, so we know that

$$\frac{\partial \mathcal{L}}{\partial A^\mu} = 0$$

we can write $(+)$ as:

$$\mathcal{L} = \partial_\nu \underbrace{\left[\partial^\mu A^\nu - \partial^\nu A^\mu \right]}$$

looks familiar?

From the form of the field eqn:

$$-\partial_\nu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)}\right]+\frac{\partial \mathcal{L}}{\partial A_\mu}=0$$

we have:

$$\text{"}\mathcal{L}=\underbrace{[\partial^\mu A^\nu-\partial^\nu A^\mu]}_{\text{antisymmetric in } \mu\leftrightarrow\nu}\partial_\nu A_\mu\text{"}$$

so keep only the antisymmetric piece of "$\partial_\nu A_\mu$" i.e.

$$\mathcal{L}=\tfrac{1}{2}[\partial^\mu A^\nu-\partial^\nu A^\mu]\cdot[\partial_\nu A_\mu-\partial_\mu A_\nu]$$

the combination $F^{\mu\nu}\equiv\partial^\mu A^\nu-\partial^\nu A^\mu$ is called the "field strength tensor"

It is made out of the "direct product" of 2 four-vecs:

$$\partial^\mu \text{ w/ } \bar\partial^\mu=\Lambda^\mu{}_\nu\partial^\nu \quad \to A^\nu \text{ w/ } \bar A^\nu=\Lambda^\nu{}_\rho A^\rho$$

$\Rightarrow$

$$\bar F^{\mu\nu}=\bar\partial^\mu\bar A^\nu-\bar\partial^\nu\bar A^\mu=\Lambda^\mu{}_\sigma\Lambda^\nu{}_\rho\,\partial^\sigma A^\rho-\Lambda^\nu{}_\rho\Lambda^\mu{}_\sigma\,\partial^\rho A^\sigma$$

$$\stackrel{!}{=}\Lambda^\mu{}_\sigma\Lambda^\nu{}_\rho F^{\sigma\rho} \quad \text{a "2}^{\text{nd}}\text{ rank tensor" transformation.}$$

Then $\mathcal{L}\propto F^{\mu\nu}F_{\mu\nu}$ is a scalar, w/ field eqn.s proportional to (+):

$$\partial_\nu[\partial^\mu A^\nu-\partial^\nu A^\mu]=0$$

By construction, $A^\mu\longrightarrow A^\mu+\partial^\mu\phi$ does not change the field eqn.s (we projected out that part), so

we can pick ϕ so that if A^μ has $\partial_\mu A^\mu=u$,

$\bar A^\mu=A^\mu+\partial^\mu\phi$ has $\partial_\mu\bar A^\mu=0$ (Lorenz gauge)

by solving:

$$\underbrace{\partial_\mu\bar A^\mu=0=\partial_\mu A^\mu+\partial_\mu\partial^\mu\phi}$$

$$u+\Box\phi=0\Rightarrow\phi=\int_{\text{all space}} G(x^\mu,x'^\mu)\,u(x'^\mu)\,d^4x'$$

Green's function integral solution

then in terms of $\bar A^\mu$, the field eqn. is:

$$\partial_\nu[\partial^\mu\bar A^\nu-\partial^\nu\bar A^\mu]=0$$

$$\text{"}$$

$$\partial^\mu\cancel{\partial_\nu\bar A^\nu}-\Box\bar A^\mu=0 \quad \text{the wave eqn. again}$$