Noether's Theorem

We showed that if you have $\mathcal{L}(\phi, \partial\phi)$, w/ ϕ satisfying:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) = 0$$

then if you have a "symmetry" of the field eqn, i.e. a transformation $\phi \to \phi + \Theta$ that leaves the field equation unchanged, there is an associated conservation law;

$$\partial_\mu J^\mu = 0 \qquad (*)$$

w/ $J^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\Theta - K^\mu \qquad (+)$

(*) says: $\frac{1}{c}\frac{\partial J^0}{\partial t} + \nabla \cdot \vec{J} = 0$ which we recognize as the "local" version of:

$$\frac{d}{dt}\int_\Omega \frac{J^0}{c}\, d\tau = -\oint \vec{J} \cdot d\vec{a}$$

We need an example to make sense of (*) & (+) - suppose the field eqn. is

$$-\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi = 0$$

this is unchanged by coordinate translation

$$x \to x + d$$

, so we'll start here

Space-Time Translation

Take $x^\mu \to x^\mu + d^\mu$ for $d^\mu \doteq \begin{pmatrix} ct_0 \\ \vec{r}_0 \end{pmatrix}$ separate, constant shifts in time & space

That should leave a field eqn. (like the wave eqn.) unchanged.

To construct the J^μ in (+), we need to know Θ & K^μ.

1. Finding Θ: $\phi(x^\mu + d^\mu) \approx \phi(x^\mu) + \underbrace{\frac{\partial \phi}{\partial x^\mu} d^\mu}_{\text{sum}}$

so $\Theta = \partial_\mu \phi\, d^\mu = \partial_\nu \phi\, d^\nu$ ← summation index relabelling.

2. Finding K^μ: the Lagrangian also depends on coord.s, so we can expand it just as we did ϕ:

$$\mathcal{L}(x^\mu + d^\mu) \approx \mathcal{L}(x^\mu) + \frac{\partial \mathcal{L}}{\partial x^\mu} d^\mu = \mathcal{L}(x^\mu) + \partial_\mu(\mathcal{L} d^\mu)$$

derivs of constant are zero.

$$K^\mu = \mathcal{L} d^\mu$$

and
$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\Theta - K^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} d^\nu \partial_\nu \phi - \mathcal{L} d^\mu$$

Kronecker: $\mathcal{L} d^\mu = \mathcal{L}\,\delta^\mu{}_\nu\, d^\nu$

$$= \underbrace{\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L}\,\delta^\mu{}_\nu\right]}_{\equiv T^\mu{}_\nu} d^\nu$$

We have $J^\mu = T^\mu{}_\nu d^\nu$ w/ $\partial_\mu J^\mu = 0$ - but this is even better than it looks:

$$\partial_\mu(T^\mu{}_\nu d^\nu) = 0 = \overset{\text{const.}}{d^\nu}\left[\partial_\mu T^\mu{}_\nu\right] = 0$$

& we can sequentially set $d^0 = \alpha$, $d^x = d^y = d^z = 0$, $d^0 = 0$, $d^x = \beta$, $d^y = d^z = 0$ etc.

then we really have:

$$\partial_\mu T^\mu{}_\nu = 0 \quad \text{for } \nu = 0,1,2,3,$$

i.e. there are four conservation laws here, one for each value of ν.

The four comes from the # of independent coord. translations we can perform. (it's the # of space-time dimensions).

Example - let's take our only $\mathcal{L}$ so far:

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu \phi\, \partial^\mu \phi$$

then $\dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi$ &

$$T^\mu{}_\nu = \partial^\mu \phi\, \partial_\nu \phi - \tfrac{1}{2}\delta^\mu{}_\nu\, \partial_\alpha \phi\, \partial^\alpha \phi$$

we have the metric for raising & lowering indices

$$\eta_{\mu\nu} \doteq \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \quad \text{w/} \quad \eta^{\mu\nu} \doteq \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

so $T^{\mu\sigma} = \eta^{\nu\sigma} T^\mu{}_\nu$

&

$$\eta^{\nu\sigma}(\partial^\mu \phi\, \partial_\nu \phi) = \partial^\mu \phi\, \partial^\sigma \phi$$

$$\eta^{\nu\sigma}\delta^\mu{}_\nu = \eta^{\mu\sigma}$$

"stress tensor"

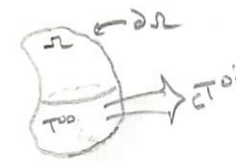
sym. in $\mu \leftrightarrow \sigma$...

then $T^{\mu\sigma} = \partial^\mu \phi\, \partial^\sigma \phi - \tfrac{1}{2}\eta^{\mu\sigma}\, \partial_\alpha \phi\, \partial^\alpha \phi = T^{\sigma\mu}$

Our 4 conservation laws, $\partial_\mu T^{\mu\sigma} = 0$, look like

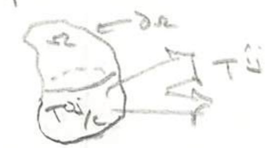
$$\partial_\mu T^{\mu 0} = 0,\quad \underbrace{\partial_\mu T^{\mu 1} = 0,\quad \partial_\mu T^{\mu 2} = 0,\quad \partial_\mu T^{\mu 3} = 0}_{\partial_\mu T^{\mu j} = 0 \quad j = 1,2,3}$$

$$\frac{1}{c}\frac{\partial T^{00}}{\partial t} + \underbrace{\frac{\partial T^{i0}}{\partial x^i}}_{\text{sum for } i=1\to 3} = 0 \quad (*)$$



& we also have, for the 3 spatial ones:

$$\frac{1}{c}\frac{\partial T^{0j}}{\partial t} + \frac{\partial T^{ij}}{\partial x^i} = 0 \quad (+)$$



observation, in $(*)$, T^{i0} appears as the "current" for T^{00}, but in $(+)$, $T^{0j}/c = T^{j0}/c$ appears as the conserved "charge"...

That sounds familiar.

What is T^{00} in terms of ϕ & its deriv.s?

$$T^{00} = \partial^0 \phi\, \partial^0 \phi - \tfrac{1}{2}\eta^{00}[\partial_\alpha \phi\, \partial^\alpha \phi]$$

$$= \left(\frac{1}{c}\frac{\partial \phi}{\partial t}\right)^2 + \frac{1}{2}\left[-\left(\frac{1}{c}\frac{\partial \phi}{\partial t}\right)^2 + \nabla\phi \cdot \nabla\phi\right]$$

$$= \frac{1}{2}\left(\frac{1}{c}\frac{\partial \phi}{\partial t}\right)^2 + \underbrace{\frac{1}{2}\nabla\phi \cdot \nabla\phi}_{\text{if } \phi \sim V,\ \text{this term looks like } \frac{1}{2}E^2 \text{ in statics...}}$$

We associate T^{00} w/ the energy (density) of the field, allowing us to identify:

$$\text{kinetic} \sim \frac{1}{2}\left(\frac{1}{c}\frac{\partial\Phi}{\partial t}\right)^2 \quad \text{potential} \sim \frac{1}{2}\nabla\Phi\cdot\nabla\Phi$$

that lends some insight into the Lagrangian:

$$\mathcal{L} = \tfrac{1}{2}\,\partial_\mu\Phi\,\partial^\mu\Phi = -\Big[\underbrace{\tfrac{1}{2}\left(\tfrac{1}{c}\tfrac{\partial\Phi}{\partial t}\right)^2}_{\text{"kinetic"}} - \underbrace{\tfrac{1}{2}\nabla\Phi\cdot\nabla\Phi}_{\text{"potential"}}\Big].$$

The identification also gives us a way to think about cT^{0i}...

And then we have also seen things like (+)

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Infinitesimal Coordinate Transformations

The translation $x^\mu \to x^\mu + a^\mu$ is the simplest coordinate transformation.

What about other ones, like rotations + Lorentz boosts?

Take rotations about the z-axis, for ex.:

$$\begin{pmatrix} c\bar{t} \\ \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\equiv \mathbb{O}_3} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

our Noetherian setup is built for infinitesimal changes to Φ: $\Phi \to \Phi + \delta\phi$ + then we Taylor expand.

But $\bar{x}^\mu = \mathbb{O}_3{}^\mu{}_\nu x^\nu$ (for rotation) is linear, but not small

we can take $\theta \ll 1$, then (o) becomes: ($\sin\theta \approx \theta$, $\cos\theta \approx 1$)

" $\bar{x} = (\mathbb{I} + \theta\mathbb{L}_z)\,x$ "

w/ $\mathbb{L}_z \doteq \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

now we have a transformation of the form:

$$\bar{x}^\mu = x^\mu + \underset{\ll 1}{\theta} L^\mu{}_\nu x^\nu \equiv x^\mu + \overset{\text{small}}{f^\mu}$$

a "mild" function of position

What changes in our development of the conserved J^μ?