Scalar Field Action

We had

$$S[\phi] = \int_\Omega \mathcal{L}(\phi, \partial\phi)\, d^4x$$

where Ω is a space-time domain, & ϕ is specified on its boundary $\partial\Omega$.

Send $\phi \to \phi + \psi$ for "small" ψ w/

$$\psi|_{\partial\Omega} = 0$$

the action responds:

$$S[\phi+\psi] = \int_\Omega \mathcal{L}(\phi+\psi, \partial\phi+\partial\psi)\, d^4x$$

$$\approx \underbrace{\int_\Omega \mathcal{L}(\phi,\partial\phi)\, d^4x}_{=S[\phi]} + \underbrace{\int_\Omega \left[\frac{\partial\mathcal{L}}{\partial\phi}\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu\psi\right] d^4x}_{\equiv \delta S}$$

we want to "extremize" the action: $\delta S = 0 \;\; \forall \psi$

but first, we have to rewrite δS in terms of ψ by itself (& not its derivs)

Div. thm. $\displaystyle\int_\Omega \partial_\mu F^\mu d^4x = \oint_{\partial\Omega} F^\mu da_\mu$

let $F^\mu \equiv \dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\psi$, then the bndry term vanishes since $\psi|_{\partial\Omega}=0$.

The product rule gives:

$$\int_\Omega \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\psi\right) d^4x = \int_\Omega \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu\psi + \int_\Omega \psi\,\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) d^4x$$

& the whole thing is $=0$, so

$$\int_\Omega \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu\psi\, d^4x = -\int_\Omega \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) d^4x$$

& we can write:

$$\delta S = \int_\Omega \left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\right]\psi\, d^4x$$

so if we require $\delta S = 0 \;\; \forall\psi$, we get the field eqn:

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = 0$$

<u>Example</u>: For

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi$$

we have $\dfrac{\partial\mathcal{L}}{\partial\phi} = 0 \qquad \dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi$

so that: $-\partial_\mu\left(\dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) + \dfrac{\partial\mathcal{L}}{\partial\phi} = 0 \Rightarrow -\partial_\mu\partial^\mu\phi = 0$

or $\Box\phi = 0$ as expected/desired

Equivalent Lagrangians

In CM, 2 Lagrangians L_1 & L_2 give the same eqn.s of motion if they differ by a total time derivative:

$$L_2 = L_1 + \frac{dk}{dt} \leftarrow k(x)$$

$$\frac{dk}{dt} = \frac{dk}{dx} \cdot \dot{x}$$

(you showed this for homework) - you can see why by returning to the action setup:

$$S_1[x] = \int_0^T L_1(x, \dot{x})\, dt$$

$$S_2[x] = \int_0^T L_1(x, \dot{x})\, dt + \int_0^T \frac{dk}{dt}\, dt$$

$$= S_1[x] + \underbrace{k(T) - k(0)}_{\text{a \#, indep. of } x(t)}$$

There's nothing new to vary (i.e. $x \to x + \epsilon f$ doesn't change this term).

Similarly, 2 field Lagrangians give the same field equations if they differ by a total divergence:

$$\mathcal{L}_2 = \mathcal{L}_1 + \partial_\mu K^\mu$$

since the action is:

$$S_2 = \int_\Omega \mathcal{L}_2\, d^4x = \int_\Omega \mathcal{L}_1\, d^4x + \int_\Omega \partial_\mu K^\mu\, d^4x$$

& $\int_\Omega \partial_\mu K^\mu\, d^4x = \oint_{\partial\Omega} K^\mu\, d\sigma_\mu$

again, a boundary term that does not contribute to the variation in the bulk.

Noether's Theorem for Fields

Suppose we have $\mathcal{L}(\phi, \partial\phi)$, & the associated field eqn.s.

Noether: If there is a field transformation

$$\phi \longrightarrow \phi + \Theta$$

that leaves the field eqn.s unchanged, then there is a "conservation law" (symmetry $\Rightarrow$ conservation)

sketch of proof: since the field eqn.s are the same, the Lagrangians $\mathcal{L}(\phi)$ & $\mathcal{L}(\phi+\Theta)$ must be related by a total divergence:

$$\mathcal{L}(\phi + \Theta) = \mathcal{L}(\phi) + \partial_\mu K^\mu \qquad (*)$$

Taylor expansion on the left gives:

$$\mathcal{L}(\phi+\Theta) \simeq \mathcal{L}(\phi) + \frac{\partial \mathcal{L}}{\partial \phi}\Theta + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\mu \Theta$$

still have $\partial\phi + \partial\Theta$ dep., just omitted for clarity

$$= \mathcal{L}(\phi) + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\Theta\right) \underbrace{- \Theta\, \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) + \Theta \frac{\partial \mathcal{L}}{\partial \phi}}_{= 0 \text{ by the field eqn.}}$$

So we have $\mathcal{L}(\phi+\Theta) \simeq \mathcal{L}(\phi) + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Theta\right)$ (+)

equating (+) & (*), we learn that:

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Theta\right) = \partial_\mu K^\mu$$

or

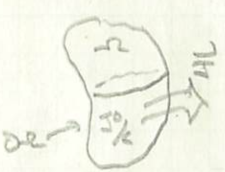
$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Theta - K^\mu\right) = 0$$

let $J^\mu \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Theta - K^\mu$

has $\partial_\mu J^\mu = 0$ — for $J^\mu \doteq \begin{pmatrix} J^0 \\ \vec{J} \end{pmatrix}$ & $\partial_\mu \doteq \begin{pmatrix} \frac{1}{c}\partial/\partial t \\ \nabla \end{pmatrix}$

$$\partial_\mu J^\mu = 0 \Rightarrow \frac{1}{c}\frac{\partial J^0}{\partial t} + \nabla\cdot\vec{J} = 0$$

or $\frac{\partial J^0/c}{\partial t} = -\nabla\cdot\vec{J}$



$$\frac{d}{dt}\int_\Omega \frac{J^0}{c}\,d\tau = -\oint_{\partial\Omega}\vec{J}\cdot d\vec{a}$$

("ordinary" 3-vol.)

J^0 is the conserved "charge" w/ $\vec{J}$ the "current"