Field Lagrangian

The action for a particle is:

$$S[x(t)] = \int_0^T L(x(t), \dot{x}(t))\, dt$$

& the extremizing Euler-Lagrange eqn. is

$$-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}(t)} + \frac{\partial L}{\partial x(t)} = 0 \quad \text{w/ } x(0)=a,\ x(T)=b \text{ given.}$$

we know that by choosing $L = \frac{1}{2}m\dot{x}(t)^2 - U(x(t))$ we recover Newton's 2^{nd} law from the E-L eqn.

We want to build an action for fields - functions of x & t, & we want the E-L eqns to give the field "eqns of motion."

Let's guess what this might look like first, then we'll work more carefully.

$$S[\phi(x,t)] = \int_0^T \int_{x_a}^{x_b} \underbrace{\mathcal{L}(\phi(x,t), \overbrace{\dot{\phi}(x,t)}^{=\frac{\partial \phi}{\partial t}}, \overbrace{\phi'(x,t)}^{=\frac{\partial \phi}{\partial x}})}_{\text{"field Lagrangian"}} \underbrace{dx\,dt}_{\text{integral over time \& space}}$$

$$\frac{\partial L}{\partial x} \longrightarrow \frac{\partial \mathcal{L}}{\partial \phi} \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \longrightarrow \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{\phi}}$$

but we should also allow for derivs w.r.t. ϕ' to count:

$$\frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial \phi'}$$

we expect to get: $-\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial \phi'} + \frac{\partial \mathcal{L}}{\partial \phi} = 0 \quad (*)$

? What is our target "eqn. of motion" here ?
From the form of (*), we expect to get 2^{nd} derivs of ϕ (just as we get 2^{nd} derivs of x in CM)

Most natural "field eqn." is $-\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} + \frac{\partial^2 \phi}{\partial x^2} = s(x,t)$

(think of $\Box V = -\rho/\epsilon_0$, $\Box \vec{A} = -\mu_0 \vec{J}$)

Then we want $-\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = -\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} \Rightarrow \mathcal{L} \sim \frac{1}{2}\frac{1}{c^2}\dot{\phi}^2$

&

$-\frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial \phi'} = \frac{\partial^2 \phi}{\partial x^2} \Rightarrow \mathcal{L} \sim -\frac{1}{2}\phi'^2$

so far, then, $\mathcal{L} = \underbrace{\frac{1}{2}(\frac{1}{c}\dot{\phi})^2}_{\text{"kinetic"}} - \frac{1}{2}(\phi')^2$

We get a "source" function from a term $\sim -\phi s$

$\mathcal{L} = \frac{1}{2}(\frac{1}{c}\dot{\phi})^2 - \frac{1}{2}(\phi')^2 - \phi \cdot s$ w/ field eqn.

$-\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} + \frac{\partial^2 \phi}{\partial x^2} - s = 0$ ✓

This structure moves naturally to 3 spatial dims

$\mathcal{L} = \frac{1}{2}(\frac{1}{c}\dot{\phi})^2 - \frac{1}{2}(\nabla\phi \cdot \nabla\phi)$ (ignore source term for now)

which is an even better starting point in 3+1-dim space-time.

Aside - Summation Conventions

Four-vectors respond to boosts & rotations "as the coordinates do"

If
$$\begin{pmatrix} c\bar{t} \\ \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & -\frac{v}{c}\gamma & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{could be a boost or a rotation...}} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

which we write:

$$\bar{x}^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu \, x^\nu$$

then any collection of 4 objects (w/ the same units) that responds to the boost like:

$$\bar{A}^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu \, x^\nu$$

here are returned by the transformation

is a 4-vector.

We can make scalars by combining vectors in sums - For a 4-vec.

$$A^\mu \doteq \begin{pmatrix} A^0 \\ A^x \\ A^y \\ A^z \end{pmatrix} \qquad \text{let } A_\mu \doteq \begin{pmatrix} -A^0 \\ A^x \\ A^y \\ A^z \end{pmatrix}$$

Make an up-index into a down one via mult. by

$$\eta_{\mu\nu} \doteq \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$ the Minkowski metric.

example $x^\mu \doteq \begin{pmatrix} ct \\ \vec{x} \end{pmatrix}$ $\quad x_\mu \doteq \begin{pmatrix} -ct \\ \vec{x} \end{pmatrix}$

you can sum over up-down pairs to get a scalar:

$$\sum_{\mu=0}^{3} x^\mu x_\mu = -(ct)^2 + \vec{x}\cdot\vec{x} \quad \text{(Minkowski length)}$$

The derivative op.: $\partial_\mu \doteq \begin{pmatrix} \frac{1}{c}\frac{\partial}{\partial t} \\ \nabla \end{pmatrix}$ has $\partial^\mu \doteq \begin{pmatrix} -\frac{1}{c}\frac{\partial}{\partial t} \\ \nabla \end{pmatrix}$

So we can make the scalar

$$\sum_{\mu=0}^{3} (\partial_\mu \phi)(\partial^\mu \phi) = -\left(\frac{1}{c}\frac{\partial \phi}{\partial t}\right)^2 + \nabla\phi\cdot\nabla\phi \quad \ldots$$

we can also make the 2nd deriv. scalar operator

$$\sum_{\mu=0}^{3} (\partial_\mu \partial^\mu)\phi = -\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi \quad \ldots$$

our "guess" Lagrangian then has the form: $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi)$

w/ "field eqn." $\partial_\mu \partial^\mu \phi = 0$ ✓.

Field Action

For $\phi(t, x, y, z)$ (which I'll denote $\phi(x)$)

$$S[\phi(x)] = \int_\Omega \underbrace{\mathcal{L}\left(\phi, \frac{1}{c}\frac{\partial \phi}{\partial t}, \nabla\phi\right)}_{\mathcal{L}(\phi, \partial\phi) \text{ shorthand}} \underbrace{d^4x}_{\text{spacetime vol.}}$$

As in the particle case, we'll assume we know $\phi \in \partial\Omega$, as a physical observation.

we'll extremize the action by taking $\phi \to \phi + \varphi$ for arbitrary (small) φ that vanishes on the boundary:

$$\varphi\big|_{\partial\Omega} = 0$$

so as not to disturb the observation, & we'll demand that $\delta S = 0 \;\forall \varphi$.

$$S[\phi + \varphi] = \int_\Omega \mathcal{L}(\phi + \varphi, \partial\phi + \partial\varphi)\, d^4x$$

$$= \underbrace{\int_\Omega \mathcal{L}(\phi, \partial\phi)\, d^4x}_{= S[\phi]} + \underbrace{\int_\Omega \left\{\left[\sum_{\mu=0}^{3} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial x^\mu}\right)} \cdot \frac{\partial\varphi}{\partial x^\mu}\right] + \frac{\partial \mathcal{L}}{\partial \phi}\varphi\right\} d^4x}_{= \delta S}$$

the term in brackets is:

$$\frac{\partial \mathcal{L}}{\partial\left(\frac{1}{c}\frac{\partial\phi}{\partial t}\right)} \frac{\partial\varphi}{c\,\partial t} + \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial x}\right)} \frac{\partial\varphi}{\partial x} + \cdots = \sum_{\mu=0}^{3} \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} \partial_\mu\varphi$$

The divergence theorem in $D = 3+1$ is:

$$\int_\Omega \sum_{\mu=0}^{3} \partial_\mu (F^\mu)\, d^4x = \oint_{\partial\Omega} \sum_{\mu=0}^{4} F^\mu\, da_\mu$$

Let $F^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\varphi$, then

$$\int_\Omega \sum_{\mu=0}^{3} \partial_\mu F^\mu = \oint \Big(\sum_{\mu=0}^{3} \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} \varphi\Big)\, da_\mu = 0$$

$\uparrow = 0$ on $\partial\Omega$

on the left, we can use the product rule:

$$\int_\Omega \sum_{\mu=0}^{3} \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} \varphi\right) d^4x = \int_\Omega \varphi \sum_{\mu=0}^{3} \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\right) d^4x$$

$$+ \int_\Omega \sum_{\mu=0}^{3} \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} \frac{\partial\varphi}{\partial x^\mu}\, d^4x = 0$$

$\uparrow$ term we have

so we can write

$$\delta S = \int_\Omega \left\{\left[\sum_{\mu=0}^{3} \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} \frac{\partial\varphi}{\partial x^\mu} + \frac{\partial \mathcal{L}}{\partial\phi}\varphi\right]\right\} d^4x$$

$$= \int_\Omega \left\{-\sum_{\mu=0}^{3} \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\right) + \frac{\partial \mathcal{L}}{\partial\phi}\right\} \varphi\, d^4x$$

& for this to be true $\forall \varphi$, we must have:

$$-\sum_{\mu=0}^{3} \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\right) + \frac{\partial \mathcal{L}}{\partial\phi} = 0$$

as we expected.