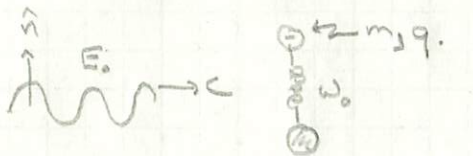
Lorentz Model



The charge (electron) moves in the $\hat{n}$ direction (we're again ignoring the magnetic contribution to the force).

$$\vec{r}(t) = u(t)\,\hat{n}$$

& the equation of motion is:

$$m\ddot{u}(t) = \underbrace{-m\omega_0^2 u(t)}_{\text{"bond" oscillation}} - \underbrace{m\tau\dddot{u}(t)}_{\text{radiation damping}} + \underbrace{qE_0 e^{i(kz(t)+ct)}}_{\text{driving force (wave)}}$$

radiation damping, $\tau \equiv \dfrac{\mu_0 q^2}{6\pi m c}$

For a particle starting from rest at the origin, $z(t)=0$
&

$$\ddot{u}(t) = -\omega_0^2 u(t) - \tau\dddot{u}(t) + \frac{qE_0}{m} e^{+i\omega t} \qquad (\omega = kc)$$

Let $u(t) = u_0 e^{+i\omega t}$ (steady state solution),

$$\dot{u}(t) = +i\omega u(t), \quad \ddot{u}(t) = -\omega^2 u(t), \quad \dddot{u}(t) = -i\omega^3 u(t)$$

& the eqn. of motion becomes:

$$\left[-\omega^2 + \omega_0^2 - i\omega^3\tau\right] u_0 e^{+i\omega t} = qE_0/m\, e^{+i\omega t}$$

$$\& \quad u_0 = \frac{qE_0/m}{\omega_0^2 - \omega^2 - i\gamma\omega} \quad \text{w/ } \gamma \equiv \omega^2\tau$$

writing $u_0 = \dfrac{qE_0/m\,[\omega_0^2 - \omega^2 + i\gamma\omega]}{(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}$ we can express $u(t)$ in polar form:

$$u(t) = \frac{qE_0/m}{[(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2]^{1/2}}\, e^{i(\omega t + \phi)} \quad \text{w/ } \phi = \tan^{-1}\left(\frac{\gamma\omega}{\omega_0^2 - \omega^2}\right)$$

so the real part is just

$$\vec{r}(t) = \frac{qE_0/m}{[(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2]^{1/2}} \cdot \cos(\omega t + \phi)\,\hat{n}$$

The associated dipole radiation has intensity:

$$\vec{I}_d = \frac{I_0 \ell^2}{r^2} \frac{\omega^4}{[(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2]} \sin^2\alpha\, \hat{r}$$

averaging over all polarizations:

$$\overline{\vec{I}}_d = \frac{\tfrac{1}{2} I_0 \frac{\ell^2}{r^2} \cdot \omega^4 (1+\cos^2\theta)\,\hat{r}}{(\omega_0^2-\omega^2)^2 + \tau^2\omega^6}$$

for $\omega_0 \to 0$, $\tau \to 0$, we recover the Thomson scattering result.

The total radiated power is:

$$P = \int_{\partial\Omega} \vec{I} \cdot d\vec{a} = \left(\frac{8\pi}{3}\right) \frac{I_0 \ell^2 \omega^4}{((\omega_0^2-\omega^2)^2 + \tau^2\omega^6)}$$

Polarization

Suppose we have a bunch of charges in a material; each has a dipole moment:

$$\vec{p}(t) = q\,\vec{r}(t) = \frac{q^2 E_0/m}{\omega_0^2-\omega^2-i\gamma\omega}\, e^{i\omega t}\,\hat{n}$$

$$= \frac{q^2/m}{\omega_0^2-\omega^2-i\gamma\omega}\,\underbrace{E_0 e^{i\omega t}\hat{n}}_{=\tilde{\vec{E}}}$$

If you have N charges/volume:

$$\tilde{\vec{P}} = \frac{Nq^2}{m}\frac{1}{[\omega_0^2-\omega^2-i\gamma\omega]}\tilde{\vec{E}} \quad (*)$$

dipole moment per unit volume.

For a linear material:

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

so now: $\tilde{\vec{P}} = \epsilon_0 \tilde{\chi}_e \tilde{\vec{E}}$

& the complex permittivity is:

$$\tilde{\epsilon} = \epsilon_0(1+\tilde{\chi}_e)$$

w/ $\tilde{\chi}_e = \frac{Nq^2}{\epsilon_0 m}\frac{1}{[\omega_0^2-\omega^2-i\gamma\omega]}$, and

$$\tilde{\epsilon} = \epsilon_0 + \frac{Nq^2}{m}\frac{1}{[\omega_0^2-\omega^2-i\gamma\omega]}$$

In this material, $\nabla^2 \tilde{\vec{E}} = \tilde{\epsilon}\mu_0 \frac{\partial^2 \tilde{\vec{E}}}{\partial t^2}$ for freq. ω

w/ $\tilde{\vec{E}} = \tilde{\vec{E}}_0 e^{i(\tilde{k}z-\omega t)}$, so

$$-\tilde{k}^2 \tilde{\vec{E}} = -\tilde{\epsilon}\mu_0\omega^2\tilde{\vec{E}} \Rightarrow \tilde{k} = \sqrt{\tilde{\epsilon}\mu_0}\,\omega = k + i\kappa$$

so we have absorption & wave-like behavior:

$$\tilde{\vec{E}} = \tilde{\vec{E}}_0 e^{-\kappa z} e^{i(kz-\omega t)}$$

damping absorbs energy, $\sim \tilde{\vec{E}}_0 \cdot \tilde{\vec{E}}_0 e^{-2\kappa z}$

The index of refraction is $n = \frac{c}{v} = \frac{ck}{\omega}$

so we need the real part, k.

"small"

$$\tilde{k} = \sqrt{\tilde{\epsilon}\mu_0}\,\omega = \omega\sqrt{\mu_0\epsilon_0}\left[1 + \frac{Nq^2}{\epsilon_0 m}\frac{1}{[\omega_0^2-\omega^2-i\gamma\omega]}\right]^{1/2}$$

$$\approx \frac{\omega}{c}\left[1 + \frac{\frac{1}{2}Nq^2}{\epsilon_0 m}\frac{1}{[\omega_0^2-\omega^2-i\gamma\omega]}\right]$$

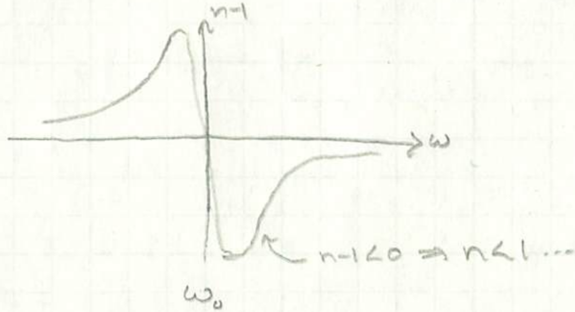
& the real part is:

$$k = \mathrm{Re}(\tilde{k}) = \frac{\omega}{c}\left[1 + \frac{\frac{1}{2}Nq^2}{\epsilon_0 m}\frac{(\omega_0^2-\omega^2)}{[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2]}\right]$$

$$\& \quad n = \frac{ck}{\omega} = 1 + \frac{Nq^2(\omega_0^2-\omega^2)}{2\epsilon_0 m[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2]}$$

the index of refraction is a function of frequency, now, so wave speed is, too; $v = c/n$ – different freq.s travel at different speeds.

a sketch of $n-1 \sim \frac{(\omega_0^2-\omega^2)}{[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2]}$



If $\omega \ll \omega_0$, the shape of n is governed by

$$n \sim \underbrace{\frac{1}{\omega_0^2-\omega^2}}_{\text{"}\gamma\approx 0\text{"}} = \frac{1}{\omega_0^2}\left(1-\frac{\omega^2}{\omega_0^2}\right)^{-1} \approx \frac{1}{\omega_0^2}\left(1+\frac{\omega^2}{\omega_0^2}\right)$$

$$n \cong 1+\left(\frac{Nq^2}{2\epsilon_0 m}\right)\left[\frac{1}{\omega_0^2}+\frac{\omega^2}{\omega_0^4}\right] = 1+A\left(1+\frac{B}{\lambda^2}\right)$$

(w/ $\lambda = 2\pi c/\omega$, the wavelength) "Cauchy's formula)

If you have a material w/ multiple molecular species, the starting point is

$$\tilde{\vec{P}} = \frac{Nq^2}{m}\left(\sum_{j=1}^{n} \frac{f_j}{\omega_j^2-\omega^2-i\omega\gamma_j}\right)\tilde{\vec{E}}$$

f_j ← # of charges w/ $\omega_0 = \omega_j$