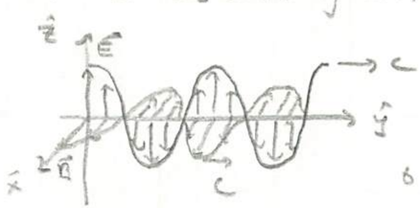
Energy / Momentum Conservation for Plane Waves

For $\vec{E} = E_0 \cos(k(y-ct))\hat{z}$, $\vec{B} = \frac{E_0}{c}\cos(k(y-ct))\hat{x}$



we have: $u = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2 = \epsilon_0 E^2$

& $\vec{E}\times\vec{B} = \frac{1}{c}E^2\hat{y} = \frac{1}{\epsilon_0 c}u\hat{y}$

the Poynting vector is $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = cu\hat{y}$, & energy conservation reads:

$$\frac{\partial u}{\partial t} = -\nabla\cdot\vec{S}$$

let's check - we'll need:

$$\frac{\partial u}{\partial t} = 2\epsilon_0 E_0^2 \cos(k(y-ct))\sin(k(y-ct))(kc)$$

$$\frac{\partial u}{\partial y} = 2\epsilon_0 E_0^2 \cos(k(y-ct))\sin(k(y-ct))(-k) = -\frac{1}{c}\frac{\partial u}{\partial t}$$

& then

$$-\nabla\cdot\vec{S} = -\frac{\partial}{\partial y}(cu) = -c\frac{\partial u}{\partial y} = \frac{\partial u}{\partial t} \checkmark$$

The momentum density is $\vec{g} = \epsilon_0\vec{E}\times\vec{B} = \frac{1}{c}u\hat{y}$ & the stress tensor has components:

$$T_{ij} = \epsilon_0\left(E_iE_j - \tfrac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \tfrac{1}{2}\delta_{ij}B^2\right)$$

w/ no non-zero off-diagonal components.

$$T_{xx} = \epsilon_0\left(-\tfrac{1}{2}E^2\right) + \frac{1}{\mu_0}\left(B^2 - \tfrac{1}{2}B^2\right) = 0$$

$$T_{yy} = \epsilon_0\left(-\tfrac{1}{2}E^2\right) + \frac{1}{\mu_0}\left(-\tfrac{1}{2}B^2\right) = -u$$

$$T_{zz} = \epsilon_0\left(E^2 - \tfrac{1}{2}E^2\right) + \frac{1}{\mu_0}\left(-\tfrac{1}{2}B^2\right) = 0.$$

& the statement of momentum conservation here is:

$$\frac{\partial\vec{g}}{\partial t} = -\nabla\cdot(-\overleftrightarrow{T}) \qquad (*)$$

recall that $(\nabla\cdot\overleftrightarrow{T})_j = \sum_{i=1}^{3}\frac{\partial T_{ij}}{\partial x^i}$ so only the $j=2$, $i=2$ terms contribute

$$\nabla\cdot\overleftrightarrow{T} = \hat{y}\left(\frac{\partial T_{yy}}{\partial y}\right) = -\hat{y}\frac{\partial u}{\partial y} = \frac{1}{c}\frac{\partial u}{\partial t}\hat{y}$$

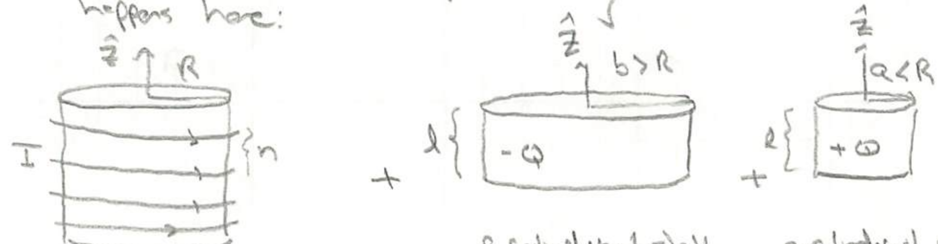
w/ $\frac{\partial\vec{g}}{\partial t} = \frac{1}{c}\frac{\partial u}{\partial t}\hat{y}$ ← match → so (*) holds here.

Angular Momentum

From the momentum density $\vec{g} = \epsilon_0\vec{E}\times\vec{B}$, we can define the angular momentum density:

$$\vec{\ell} = \vec{r}\times\vec{g} = \epsilon_0\vec{r}\times(\vec{E}\times\vec{B})$$

let's set up an example configuration to see what happens here:



infinite solenoid of radius R w/ n turns of wire per unit length carrying steady current I, $\vec{B} = \mu_0 nI\hat{z}$ inside the solenoid

a cylindrical shell of radius $b>R$ & length ℓ w/ charge $-Q$ spread uniformly around its surface.

a cylindrical shell of radius $a<R$, length ℓ, w/ charge $+Q$ spread uniformly around its surface,

Assuming the cylinders are long, the electric field due to the pair is confined to the region $a < s < b$:

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{s} \qquad (\lambda = Q/\ell)$$

& the solenoid provides

$$\vec{B} = \mu_0 n I \hat{z} \quad \text{for } 0 < s < R$$

the momentum density is:

$$\vec{g} = \epsilon_0 \vec{E}\times\vec{B} = -\frac{\mu_0 n I \lambda}{2\pi s}\hat{\phi}$$

and $\vec{\ell} = \underset{= s\hat{s}}{\vec{r}} \times \vec{g} = -\frac{\mu_0 n I \lambda}{2\pi}\hat{z}$

This $\vec{\ell}$ is only non-zero for $a < s < R$, where both $\vec{E}$ & $\vec{B}$ are non-zero.

The total angular momentum stored in the configuration is:

$$\vec{L} = \int_{\mathcal{V}} \vec{\ell}\, d\tau = -\frac{\mu_0 n I \lambda}{2\pi}\underbrace{\pi(R^2 - a^2)\ell}_{\text{vol.}}\hat{z}$$

$$= -\tfrac{1}{2}\mu_0 n I Q (R^2 - a^2)\hat{z}$$

Now suppose you turn off the current in the solenoid, so that I is a function of t (while turning it off)

There is an induced electric field:

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \Rightarrow \oint_{\partial S}\vec{E}\cdot d\vec{\ell} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{a} \qquad (0)$$

For a surface S that is a disk of radius s & assuming $\vec{E} = E(s)\hat{\phi}$ for a magnetic source: we have

$$\oint \vec{E}\cdot d\vec{\ell} = E(s)\cdot 2\pi s \quad \& \quad -\int_{\partial S}\dot{\vec{B}}\cdot d\vec{a} = \begin{cases} -\mu_0 n \dot{I}\,\pi s^2 & s \le R \\ -\mu_0 n \dot{I}\,\pi R^2 & s \ge R \end{cases}$$

& putting the two sides of (0) together, we get:

$$\vec{E} = \begin{cases} -\tfrac{1}{2}\mu_0 n \dot{I}\cdot s\,\hat{\phi} & s < R \\ -\tfrac{1}{2}\mu_0 n \dot{I}\, R^2/s\,\hat{\phi} & s \ge R \end{cases}$$

this electric field turns the charged cylinders.

For the $+Q$ cylinder, the torque due to $\vec{E}$ is

$$\vec{\tau}_+ = \underset{= s\hat{s}}{\vec{r}} \times (Q\vec{E}) = -\tfrac{1}{2}\mu_0 n \dot{I} Q\cdot a^2\hat{z}$$

then Newton's 2nd law reads:

$$\frac{d\vec{L}_+}{dt} = \vec{\tau}_+ \Rightarrow \vec{L}_+ = \int_0^t \vec{\tau}_+\, dt = -\tfrac{1}{2}\mu_0 n Q a^2 \int_I^0 dI\,\hat{z}$$

$$= \tfrac{1}{2}\mu_0 n Q I a^2\hat{z}$$

For the outer cylinder,

$$\vec{T}_- = \vec{r} \times (-Q\vec{E}) = +\tfrac{1}{2}\mu_0 n \dot{I} R^2 \hat{z}$$

$s\hat{s}$

w/ $\frac{d\vec{L}_-}{dt} = \vec{T}_- \Rightarrow \vec{L}_- = \int_0^t \vec{T}_- \, dt = -\tfrac{1}{2}\mu_0 n Q I R^2 \hat{z}$

the total angular momentum of the cylinders after the current has been turned off is:

$$\vec{L} = \vec{L}_+ + \vec{L}_- = -\tfrac{1}{2}\mu_0 n Q I (R^2 - a^2)\hat{z}$$

precisely the amount stored, initially, in the fields.

Radiation

Just as radiation carries energy away from the configuration of moving charge that generates it according to

$$\frac{d}{dt}\int_\Omega u \, d\tau = -\oint_{\partial\Omega} \vec{S} \cdot d\vec{a}$$

which gave us the Larmor formula,

so, too, does that radiation remove momentum & angular momentum from the source configuration.

you can use $\frac{d}{dt}\int_\Omega \vec{g} \, d\tau = -\oint_{\partial\Omega} (-\overleftrightarrow{T}) \cdot d\vec{a}$

for a spherical surface $\partial\Omega$ out at infinity to find the "momentum radiated away" - ends up being:

$$\frac{\mu_0 q^2 a^2}{6\pi c^3}\vec{v} \quad \text{(momentum radiated per unit time).}$$

For angular momentum - we can take

$\frac{\partial \vec{g}}{\partial t} = \nabla \cdot \overleftrightarrow{T}$ & cross $\vec{r}$ into both sides:

$$\frac{\partial}{\partial t}(\vec{r} \times \vec{g}) = \vec{r} \times (\nabla \cdot \overleftrightarrow{T}) \overset{?}{=} \nabla \cdot (\vec{r} \times \overleftrightarrow{T})$$

vector identity

& integrating both sides over a "large" spherical domain Ω

$$\frac{d}{dt}\int_\Omega \vec{l} \, d\tau = -\oint_{\partial\Omega} (\vec{r} \times (-\overleftrightarrow{T})) \cdot d\vec{a}$$

from which you can extract the angular momentum radiated per unit time.

$$\frac{\mu_0 q^2}{6\pi c}(\vec{v} \times \vec{a}).$$