Momentum Conservation

For a domain Ω filled w/ moving charge generating & responding to $\vec E$ & $\vec B$ fields, we had

$$\frac{d\vec P_{mech}}{dt} = -\frac{d}{dt}\underbrace{\int_\Omega \vec g\, d\tau}_{\text{momentum in the fields}} - \oint_{\partial\Omega}(-\overleftrightarrow{T})\cdot d\vec a \quad (*)$$

w/ $\vec g \equiv \epsilon_0 \vec E\times\vec B$ the momentum density

& $\overleftrightarrow{T}$ w/ entries: $T_{ij} = \epsilon_0\left(E_iE_j - \frac12\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac12\delta_{ij}B^2\right)$
the "Maxwell stress tensor"

This tensor has divergence given by:

$$\sum_{i=1}^{3} \frac{\partial T_{ij}}{\partial x^i} = \sum_{i=1}^{3} \epsilon_0\left(\frac{\partial E_i}{\partial x^i}E_j + E_i\frac{\partial E_j}{\partial x^i} - \frac12\delta_{ij}\frac{\partial(E^2)}{\partial x^i}\right) + \frac{1}{\mu_0}(E\to B) \qquad \text{for } j=1,2,3.$$

$$= \epsilon_0\left[(\nabla\cdot\vec E)E_j + (\vec E\cdot\nabla)E_j - \frac12\frac{\partial(E^2)}{\partial x^j}\right] + \frac{1}{\mu_0}[E\to B]$$

for $j=1,2,3$.

These 3 eqn.s give the 3 components of $\nabla\cdot\overleftrightarrow{T}$:

$$(\nabla\cdot\overleftrightarrow{T})_j = \epsilon_0\left[(\nabla\cdot\vec E)E_j + (\vec E\cdot\nabla)E_j - \frac12\frac{\partial(E^2)}{\partial x^j}\right] + \frac{1}{\mu_0}[E\to B]$$

And if $A_j = B_j$ for $j=1,2,3$, $\vec A = \vec B$, so

$$\nabla\cdot\overleftrightarrow{T} = \epsilon_0\left[(\nabla\cdot\vec E)\vec E + (\vec E\cdot\nabla)\vec E - \frac12\nabla E^2\right] + \frac{1}{\mu_0}\left[(\nabla\cdot\vec B)\vec B + (\vec B\cdot\nabla)\vec B - \frac12\nabla B^2\right]$$

and we had force density in Ω:

$$\vec f = \nabla\cdot\overleftrightarrow{T} - \frac{\partial\vec g}{\partial t}. \qquad (+)$$

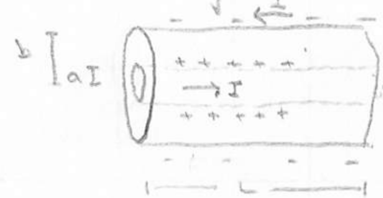
Integrating over Ω gave

$$\vec F = -\frac{d}{dt}\int_\Omega \vec g\, d\tau - \int_\Omega(-\nabla\cdot\overleftrightarrow{T})\, d\tau$$

which returns $(*)$ using the divergence theorem on the 2nd term.

Examples & Interpretation

Let's find the total momentum stored in a simple field configuration:



$\vec E = 0$ & $\vec B = 0$ for $s>b$ & $s<a$. In between, for $a<s<b$,

$$\vec E = \frac{\lambda}{2\pi\epsilon_0 s}\hat s \qquad \vec B = \frac{\mu_0 I}{2\pi s}\hat\phi$$

the momentum density is: $\vec g = \epsilon_0\vec E\times\vec B = \frac{\mu_0 I\lambda}{4\pi^2 s^2}\hat z$

The total momentum in a segment of length L is:

$$\vec P_{em} = \int_\Omega \vec g\, d\tau = \int_0^L\int_0^{2\pi}\int_a^b \frac{\mu_0 I\lambda}{4\pi^2 s^2}\hat z\cdot s\, ds\, d\phi\, dz$$

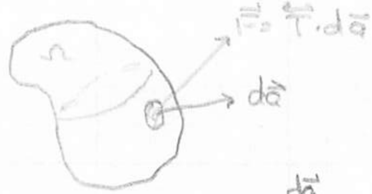
$$= \frac{\mu_0 I\lambda L}{2\pi}\log\left(b/a\right)\hat z$$

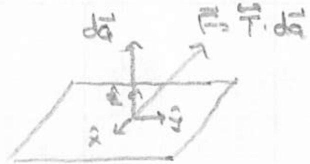
If you turned off the fields, this is the momentum that would be imparted to the charges.

How about the stress tensor, what is that? For static fields, the net force on charges in Ω is, from (*):

$$\vec{F} = \oint_{\partial\Omega} \overleftrightarrow{T}\cdot d\vec{a}$$

so $\overleftrightarrow{T}\cdot d\vec{a}$ is the forcing acting on a patch of the surface w/ area elt. $d\vec{a}$





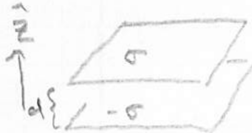
For a patch w/ $d\vec{a} = da\,\hat{z}$: we have

$$\begin{pmatrix} F^x \\ F^y \\ F^z \end{pmatrix} = \begin{pmatrix} T^{xx} & T^{xy} & T^{xz} \\ T^{yx} & T^{yy} & T^{yz} \\ T^{zx} & T^{zy} & T^{zz} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ da \end{pmatrix} = \begin{pmatrix} T^{xz} \\ T^{yz} \\ T^{zz} \end{pmatrix} da$$

& we can think of the col.s of $\overleftrightarrow{T}$ as the force per unit area acting on a surface w/ $\hat{n} = \hat{x}$ (1st col.), $\hat{n} = \hat{y}$ (2nd col.), $\hat{n} = \hat{z}$ (3rd col)

$$\overleftrightarrow{T} \doteq \left(\vec{F}^1/A \;\middle|\; \vec{F}^2/A \;\middle|\; \vec{F}^3/A \right)$$

Take:



$\vec{E} = 0$ for $z<0$, $z>d$, in between, $\vec{E} = -\sigma/\epsilon_0\,\hat{z}$

For a patch of area A on the top sheet, the force per unit area is:

$$\vec{F} = \underbrace{(\sigma A)}_{\text{"}q\text{"}}\underbrace{\tfrac{1}{2}}_{\text{avg. of inside \& outside fields}}(-\sigma/\epsilon_0\,\hat{z}) = -\tfrac{1}{2}\sigma^2 A/\epsilon_0\,\hat{z} \Rightarrow \vec{F}/A = -\tfrac{1}{2}\,\sigma^2/\epsilon_0\,\hat{z}$$

? What do we expect for $\overleftrightarrow{T}$?

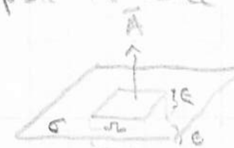
$T_{ij} = \epsilon_0 (E_i E_j - \tfrac{1}{2}\delta_{ij} E^2)$ (no $\vec{B}$ here)

$T_{xx} = -\tfrac{1}{2}\epsilon_0 E^2$ $\quad T_{xy} = T_{yx} = T_{xz} = T_{zx} = T_{yz} = T_{zy} = 0$

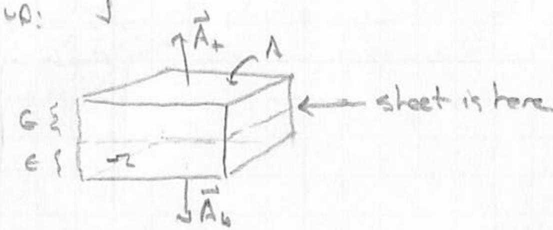
$T_{yy} = -\tfrac{1}{2}\epsilon_0 E^2$, $T_{zz} = \tfrac{1}{2}\epsilon_0 E^2$

or $T_{xx} = T_{yy} = -\tfrac{1}{2}\sigma^2/\epsilon_0$ & $T_{zz} = \tfrac{1}{2}\sigma^2/\epsilon_0$

let's compute the force on a patch of area A of the top sheet:



w/ need to compute: $\oint \overleftrightarrow{T}\cdot d\vec{a}$ for the pillbox shown above, & blown up:



Even though T_{xx} & T_{yy} are non-zero, they integrate to zero around the side of the pillbox:

$$\int_{\substack{\text{front} \\ \text{w/ } \hat{n}=\hat{x}}} T_{xx}\,da + \int_{\substack{\text{back} \\ \text{w/ } \hat{n}=-\hat{x}}} T_{xx}(-da) + \int_{\substack{\text{left} \\ \text{w/ } \hat{n}=-\hat{y}}} T_{yy}(-da) + \int_{\substack{\text{right} \\ \text{w/ } \hat{n}=\hat{y}}} T_{yy}\,da = 0$$

so only the top & bottom surfaces contribute, & for the top surface, $T_{ij}=0$ — there is no field above the top sheet of charge, & for the bottom surface, w/ $\hat{n}=-\hat{z}$, we have

$$\int T_{zz}(-da) = -\tfrac{1}{2}\sigma^2/\epsilon_0$$

so all in all: $\vec{F} = \oint \overleftrightarrow{T}\cdot d\vec{a} = -\tfrac{1}{2}\sigma^2/\epsilon_0\,\hat{z}$ ✓

Momentum Conservation in Charge-Free Regions

In regions where there is no charge, so that $\frac{d\vec{p}_{\text{mech}}}{dt}=0$, (*) gives

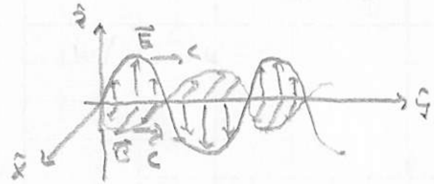
$$\frac{d}{dt}\int_\Omega \vec{g}\,d\tau = -\oint_{\partial\Omega} (-\overleftrightarrow{T})\cdot d\vec{a} \quad (+)$$

w/ $\overleftrightarrow{T}$ acting as the "momentum current density"
since (+) must hold for all Ω, we have the local:

$$\frac{\partial\vec{g}}{\partial t} = -\nabla\cdot(-\overleftrightarrow{T})$$ (just (+) w/ $\vec{p}=0$ since there are no charges in Ω).

Let's compute $\vec{g}$ & $\overleftrightarrow{T}$ for a typical plane wave setup:

$\vec{E} = E_0\cos(k(y-ct))\,\hat{z}$, $\vec{B} = E_0/c\,\cos(k(y-ct))\,\hat{x} = E/c\,\hat{x}$



$$u = \tfrac{1}{2}\epsilon_0 E^2 + \tfrac{1}{2\mu_0}B^2 = \epsilon_0 E^2 = \epsilon_0 E_0^2\cos^2(k(y-ct))$$

$$\vec{E}\times\vec{B} = \tfrac{1}{c}E^2\hat{y} = \tfrac{1}{c}E_0^2\cos^2(k(y-ct))\,\hat{y}$$

& $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0 c}E^2\hat{y}$

energy conservation is $\frac{\partial u}{\partial t} = -\nabla\cdot\vec{S}$

$\vec{g} = \epsilon_0\vec{E}\times\vec{B} = \frac{\epsilon_0}{c}E^2\hat{y}$, & it is interesting to note that $\vec{S}\sim\vec{E}\times\vec{B}$ is an energy current density, while $\vec{g}\sim\vec{E}\times\vec{B}$ is itself a conserved density...

$$T_{ij} = \epsilon_0\left(E_iE_j - \tfrac{1}{2}\delta_{ij}E^2\right) + \tfrac{1}{\mu_0}\left(B_iB_j - \tfrac{1}{2}\delta_{ij}B^2\right)$$

so

$$T_{xx} = -\tfrac{1}{2}\epsilon_0E^2 + \tfrac{1}{2}\tfrac{1}{\mu_0}\tfrac{E^2}{c^2} = 0$$

$$T_{yy} = -\tfrac{1}{2}\epsilon_0E^2 - \tfrac{1}{2\mu_0}\tfrac{E^2}{c^2} = -\epsilon_0E^2 = -u$$ all other T_{ij} are zero.

$$T_{zz} = \tfrac{1}{2}\epsilon_0E^2 - \tfrac{1}{2}\tfrac{1}{\mu_0}\tfrac{E^2}{c^2} = 0$$

$\vec{g} = \frac{1}{c}u\hat{y}$, $T_{yy} = -u$,

w/ $\frac{\partial u}{\partial t} = 2\epsilon_0E_0^2\cos(k(y-ct))\sin(k(y-ct))\cdot kc$

$\frac{\partial u}{\partial y} = -2\epsilon_0E_0^2\cos(k(y-ct))\sin(k(y-ct))\cdot k = -\frac{1}{c}\frac{\partial u}{\partial t}$

so $\frac{\partial\vec{g}}{\partial t} = \frac{1}{c}\frac{\partial u}{\partial t}\hat{y}$ & $\nabla\cdot\overleftrightarrow{T} = \frac{\partial T_{yy}}{\partial y}\hat{y} = +\frac{1}{c}\frac{\partial u}{\partial t}\hat{y}$

$\to \frac{\partial\vec{g}}{\partial t} = -(-\nabla\cdot\overleftrightarrow{T})$ ✓