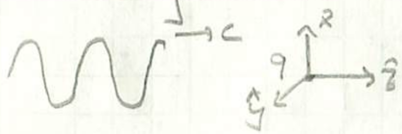
Thompson Scattering

Setup: 

plane wave hits a free charge q, what is the radiated field?

$\vec{E} = E_0 \cos(k(z-ct))\hat{x}$, $\vec{B} = \frac{E_0}{c}\cos(k(z-ct))\hat{y}$

The charge q is initially at rest at the origin, & we need to solve

$$m\ddot{\vec{r}}(t) = q\vec{E} + q\vec{v}\times\vec{B}.$$

As in the past, we'll take the electric force as the dominant one:

$$m\ddot{x}(t) = qE_0\cos(k(z-ct)), \quad \ddot{y}(t)=0, \quad \ddot{z}(t)=0$$

w/ $x(0)=y(0)=z(0)=0$, $\dot{x}(0)=\dot{y}(0)=\dot{z}(0)=0$.

$y(t)=0$ & $z(t)=0$ solve Newton's 2nd law, w/ i.c.s.

Then $\ddot{x}(t) = \frac{qE_0}{m}\cos(kct) \Rightarrow x(t) = A - \frac{qE_0}{mk^2c^2}\cos(kct)$
& $\dot{x}(0)=0$ automatically, but

$$x(0) = A - \frac{qE_0}{mk^2c^2} = 0 \Rightarrow A = +\frac{qE_0}{mk^2c^2}, \text{ \&}$$

$$x(t) = \frac{qE_0}{mk^2c^2}\left(1-\cos(kct)\right)$$

Now we have an oscillating dipole, $\vec{p} = qx(t)\hat{x}$, & we know that the radiation field is:

$$\vec{B}_d = -\frac{\mu_0}{4\pi rc}\left(\hat{r}\times\ddot{\vec{p}}(t_r)\right) \quad \text{w/ } t_r = t - r/c$$

$$\ddot{\vec{p}}(t) = q\ddot{x}(t)\hat{x} = \frac{q^2E_0}{m}\cos(kct_r)\left[\sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}\right]$$

so

$$\vec{B}_d = -\frac{\mu_0 q^2 E_0}{4\pi mrc}\cos(kct_r)\left[\cos\theta\cos\phi\,\hat{\phi} + \sin\phi\,\hat{\theta}\right]$$

$$\vec{E}_d = -c\hat{r}\times\vec{B}_d = \frac{\mu_0 q^2 E_0}{4\pi mr}\cos(kct_r)\left[-\cos\theta\cos\phi\,\hat{\theta} + \sin\phi\,\hat{\phi}\right]$$

The Poynting vector is

$$\vec{S}_d = \frac{1}{\mu_0}\vec{E}_d\times\vec{B}_d = \frac{\mu_0}{c}\left(\frac{q^2E_0}{4\pi m}\frac{\cos(kct_r)}{r}\right)^2\left[\cos^2\theta\cos^2\phi + \sin^2\phi\right]\hat{r}$$

From the expression for $\vec{E}_d$, we see that $\ell \equiv \frac{\mu_0 q^2}{4\pi m}$ is a length scale, w/

$$\vec{S}_d = \frac{\ell^2}{\mu_0 c}\frac{E_0^2\cos^2(kct_r)}{r^2}\left[\cos^2\theta\cos^2\phi + \sin^2\phi\right]\hat{r}$$

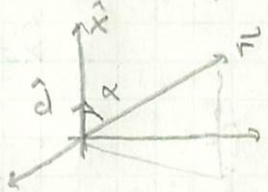
the intensity is the time average of $\vec{S}$ (just throw in a ½)

$$\vec{I}_d = \langle\vec{S}_d\rangle = \frac{1}{2}\frac{\ell^2}{\mu_0 c}\frac{E_0^2}{r^2}\left[\cos^2\theta\cos^2\phi + \sin^2\phi\right]\hat{r}$$

The incoming plane wave has $\vec{I}_0 = \frac{1}{2}\frac{E_0^2}{\mu_0 c}\hat{z}$, so we can write:

$$\vec{I}_d = I_0 \frac{l^2}{r^2}\left(\cos^2\theta\cos^2\phi + \sin^2\phi\right)\hat{r}$$

We can also use the angle between the dipole: $\hat{x}$ & the field point, $\hat{r}$, to write $\vec{I}_d$:



$$\hat{x}\cdot\hat{r} = \sin\theta\cos\phi \equiv \cos\alpha$$

so
$$\sin^2\alpha = 1-\sin^2\theta\cos^2\phi = 1-(1-\cos^2\theta)\cos^2\phi = \underbrace{1-\cos^2\phi}_{=\sin^2\phi} + \cos^2\theta\cos^2\phi$$

$$\vec{I}_d = I_0\frac{l^2}{r^2}\sin^2\alpha\,\hat{r}$$

The total power passing through a sphere of radius R is:

$$P = \int_{\partial\Omega}\vec{I}_d\cdot d\vec{a} = \int_0^{2\pi}\int_0^{\pi} I_0\frac{l^2}{R^2}\sin^2\alpha\cdot R^2\sin\theta\, d\theta\, d\phi = \int_0^{2\pi}\int_0^{\pi} I_0 l^2\sin^2\alpha\, \underbrace{\sin\theta\, d\theta\, d\phi}_{d\Omega}$$

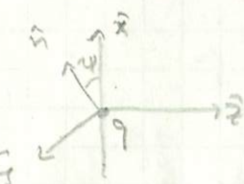
people refer to the "non-$d\Omega$" piece of the integrand as

$$\frac{dP}{d\Omega} \equiv I_0 l^2\sin^2\alpha = I_0 l^2\left[\cos^2\theta\cos^2\phi + \sin^2\phi\right]$$

the "differential cross section"

General Polarization

Take: $\vec{E} = E_0\cos(k(z-ct))\underbrace{\left[\cos\psi\,\hat{x} + \sin\psi\,\hat{y}\right]}_{\equiv\hat{n}}$



now the charge at the origin has $\vec{p}(t) = \frac{q^2 E_0}{m k^2 c^2}\cos(kct)\,\hat{n}$ &

$$\vec{B}_d = -\frac{\mu_0}{4\pi r c}\hat{r}\times\ddot{\vec{p}}(t_r) = -\frac{\mu_0 q^2 E_0}{4\pi m r c}\cos(kct_r)(\hat{r}\times\hat{n}) = -\frac{E_0 l}{rc}\cos(kct_r)\,\hat{r}\times\hat{n}$$

$$\vec{E}_d = -c\,\hat{r}\times\vec{B}_d = \frac{E_0 l}{r}\cos(kct_r)\,\hat{r}\times(\hat{r}\times\hat{n})$$

The Poynting vector is:

$$\vec{S} = \frac{1}{\mu_0}\vec{E}_d\times\vec{B}_d = -\frac{E_0^2}{\mu_0 c}\frac{l^2}{r^2}\cos^2(kct_r)\left[\hat{r}\times(\hat{r}\times\hat{n})\right]\times\overbrace{(\hat{r}\times\hat{n})}^{\equiv\vec{A} = \sin\alpha\,\hat{A}}$$

α angle between $\hat{n}$ & $\hat{r}$.

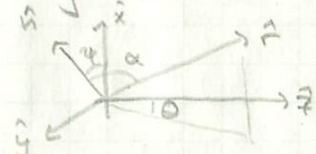
& we need: $[\hat{r}\times\vec{A}]\times\vec{A} = -\vec{A}\times[\hat{r}\times\vec{A}] = -\left\{\hat{r}\,(\underbrace{\vec{A}\cdot\vec{A}}_{=\sin^2\alpha}) - \vec{A}\,(\cancel{\hat{r}\cdot\vec{A}})\right\}$

$$= \frac{E_0^2}{\mu_0 c}\frac{l^2}{r^2}\cos^2(kct_r)\sin^2\alpha\,\hat{r}$$

The intensity is:

$$\vec{I}_d = \langle\vec{S}\rangle = I_0\frac{l^2}{r^2}\sin^2\alpha\,\hat{r}$$

To get back to spherical coord.s,



$$\hat{n}\cdot\hat{r} = \cos\alpha$$
$$= \cos\psi\,\frac{x}{r} + \sin\psi\,\frac{y}{r}$$
$$= \cos\psi\cdot\sin\theta\cos\phi + \sin\psi\sin\theta\sin\phi$$
$$= \sin\theta\left(\cos(\phi-\psi)\right)$$

& then
$$\sin^2\alpha = 1-\cos^2\alpha = 1-\sin^2\theta\cos^2(\phi-\psi).$$

the intensity is:
$$\bar{I}_\alpha = I_0\,\frac{\ell^2}{r^2}\left[1-\sin^2\theta\cos^2(\phi-\psi)\right]\hat{r}$$

& averaging over all polarizations:
$$\bar{\bar{I}}_\alpha = \frac{1}{2\pi}\int_0^{2\pi}\bar{I}_\alpha\,d\psi = \frac{I_0\ell^2}{r^2}\underbrace{\left[1-\tfrac{1}{2}\sin^2\theta\right]}_{=\frac{1}{2}+\frac{1}{2}-\frac{1}{2}\cos^2\theta}\hat{r}$$
$$= \frac{I_0\ell^2}{2r^2}\left(1+\cos^2\theta\right)\hat{r}$$

It is interesting that this result is missing the ω^4 dependence we have come to expect from electric dipole radiation intensity...