Conductors

For an "imperfect" conductor:

$$\vec{J} = \sigma \vec{E} \qquad \text{Ohm's Law}$$

σ conductivity ($\sigma \to \infty$ for perfect conductor).

Suppose you started out w/ ρ_0 inside a conductor. Charge conservation says:

$$\frac{\partial \rho}{\partial t} = -\nabla\cdot\vec{J} = -\nabla\cdot(\sigma\vec{E})$$

+ $\nabla\cdot\vec{E} = \rho/\epsilon$ ← linear material,

$$= -\sigma/\epsilon\, \rho \Rightarrow \rho(t) = \rho_0 e^{-\sigma t/\epsilon}$$

the charge dissipates w/ timescale $\tau \equiv \epsilon/\sigma$ ($\tau \xrightarrow[\sigma\to\infty]{} 0$).

Maxwell's eqns in a linear conducting medium read:

$$\nabla\cdot\vec{E} = 0 \quad \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad \nabla\cdot\vec{B} = 0$$

and: ($t \gg \tau$)

$$\nabla\times\vec{B} = \mu\sigma\vec{E} + \mu\epsilon\frac{\partial\vec{E}}{\partial t}.$$

Taking the curl of $\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ gives:

$$\nabla(\nabla\cdot\vec{E}) - \nabla^2\vec{E} = -\frac{\partial}{\partial t}\left(\mu\sigma\vec{E} + \mu\epsilon\frac{\partial\vec{E}}{\partial t}\right)$$

the modified wave eqn.

$$-\mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} + \nabla^2\vec{E} - \mu\sigma\frac{\partial\vec{E}}{\partial t} = 0 \quad + \text{sim. for } \vec{B}. \qquad (*)$$

There are plane wave solutions here:

$$\tilde{\vec{E}} = \tilde{\vec{E}}_0 e^{i(\tilde{k}z - \omega t)}$$

but now: $\frac{\partial\tilde{\vec{E}}}{\partial t} = -i\omega\tilde{\vec{E}} \qquad \frac{\partial^2\tilde{\vec{E}}}{\partial t^2} = -\omega^2\tilde{\vec{E}}$

$$\nabla^2\tilde{\vec{E}} = -\tilde{k}^2\tilde{\vec{E}}$$

so that (*) becomes:

$$\left[-\mu\epsilon(-\omega^2) - \tilde{k}^2 + i\omega\mu\sigma\right]\tilde{\vec{E}} = 0 \Rightarrow \tilde{k}^2 = \mu\epsilon\omega^2 + i\omega\mu\sigma$$

For $\tilde{k} = k + i\kappa$, $\tilde{k}^2 = k^2 + 2ik\kappa - \kappa^2 = \mu\epsilon\omega^2 + i\omega\mu\sigma$ gives:

$$k^2 - \kappa^2 = \mu\epsilon\omega^2 \quad + \quad 2k\kappa = \omega\mu\sigma$$

and we can solve for k + κ separately:

$$k = \omega\sqrt{\frac{\epsilon\mu}{2}}\left(1 + \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2}\right)^{1/2}$$

$$\kappa = \omega\sqrt{\frac{\epsilon\mu}{2}}\left(-1 + \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2}\right)^{1/2}.$$

then $\tilde{\vec{E}} = \tilde{\vec{E}}_0 e^{i(kz - \omega t)} \cdot \underbrace{e^{-\kappa z}}_{\text{decaying exponential in material}}$

the "skin depth" is $d \equiv 1/\kappa$.

all the wave parameters are associated w/ the real part, k:

$\lambda = \frac{2\pi}{k}$ $\quad v = \frac{\omega}{k}$ $\quad n = \frac{ck}{\omega}$

$\nabla\cdot\vec{E} = 0$, $\nabla\cdot\vec{B} = 0 \Rightarrow \vec{E}, \vec{B} \perp \vec{k}$

these are still transverse waves.

Take $\vec{E} = \tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)}\hat{x}$, then

$\vec{B} = \tilde{k}\frac{\tilde{E}_0}{\omega} e^{-\kappa z} e^{i(kz-\omega t)}\hat{y}$

the major difference here, aside from the exponential decay, is the phase relation between $\vec{E}$ & $\vec{B}$.

$\tilde{k} = k + i\kappa = K e^{i\phi}$ w/ $K = \sqrt{k^2+\kappa^2}$

&

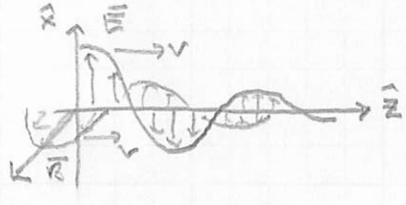
$\phi = \tan^{-1}(\kappa/k)$, so

$\tilde{B}_0 \equiv \frac{\tilde{k}\tilde{E}_0}{\omega} = \frac{K}{\omega}\tilde{E}_0 e^{i\phi}$

so $\vec{E}$ & $\vec{B}$ are out of phase:

$\vec{E} = E_0 e^{-\kappa z}\cos(kz-\omega t)\hat{x}$

$\vec{B} = B_0 e^{-\kappa z}\cos(kz-\omega t+\phi)\hat{y}$.



out of phase w/ decaying amplitude.

Dispersion

In many materials, wave speed is determined by frequency.

Suppose you have 2 waves w/ different frequencies

$w_1 = Ae^{i(k_1 z-\omega_1 t)}$ $\quad w_2 = Ae^{i(k_2 z-\omega_2 t)}$

& w_1 has speed $v_1 = \omega_1/k_1$, w_2 has $v_2 = \omega_2/k_2$

The superposition of these waves is:

$\tilde{w} = w_1 + w_2 = A[e^{i(k_1 z-\omega_1 t)} + e^{i(k_2 z-\omega_2 t)}]$

let $k \equiv \frac{1}{2}(k_1+k_2)$, $\omega \equiv \frac{1}{2}(\omega_1+\omega_2)$

&

$\Delta k \equiv \frac{1}{2}(k_1-k_2)$ $\quad \Delta\omega \equiv \frac{1}{2}(\omega_1-\omega_2)$

w/ $k+\Delta k = k_1$, $k-\Delta k = k_2$ & sim. for $\omega \pm \Delta\omega$

We can write: $e^{i(k_1 z-\omega_1 t)} = e^{i(k-\omega t)} e^{i(\Delta k-\Delta\omega t)}$

& $e^{i(k_2 z-\omega_2 t)} = e^{i(k-\omega t)} e^{-i(\Delta k-\Delta\omega t)}$

then the sum is:

$$\tilde{w} = 2Ae^{i(kz-\omega t)}\cos(\Delta k z - \Delta\omega t)$$

or, taking the imaginary part:

$$w = 2A\sin(kz-\omega t)\cos(\Delta k z - \Delta \omega t)$$

a product of two waves - one has wavelength

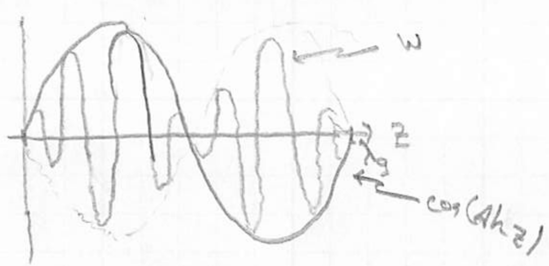
$\lambda_p = \frac{2\pi}{k}$ & travels at speed $v_p = \omega/k$

the other has wavelength

$\lambda_g = \frac{2\pi}{\Delta k}$ & travels at speed $v_g = \frac{\Delta\omega}{\Delta k}$.

notice that $\lambda_g > \lambda_p$, forming an "envelope" that travels at v_g.

At $t=0$, $w = \underbrace{[2A\cos(\Delta k z)]}_{\text{amplitude of}} \sin(kz)$



the oscillation inside the envelope travels at $v_p = \frac{\omega}{k}$.

For ω a continuous function of k, $\omega(k)$, the "phase velocity" is $v_p = \omega/k$, the "group velocity" is

$$v_g = \frac{\Delta\omega}{\Delta k} \longrightarrow \frac{d\omega}{dk}.$$

we have always had $\omega = kv$, so the phase & group velocities are the same.