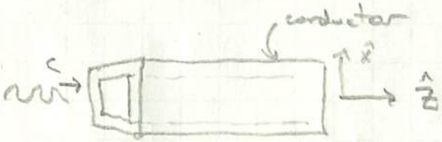
Wave Guides



a wave propagates in a hollow conductor.

$\vec{E} = \vec{E}_0 e^{i(kz-\omega t)}$ $\quad\vec{B} = \vec{B}_0 e^{i(kz-\omega t)}$ $\quad(*)$ (no relation between k & ω yet)

Inside the conductor, $\vec{E}=0$ & $\vec{B}=0$, & the continuous b.c. for the electric & magnetic fields give:

$\vec{E}^{\parallel} = 0 \qquad B^{\perp}=0$

at the interior surface (there could be surface charge/current).

In order to satisfy the b.c., we let $\vec{E}_0$ & $\vec{B}_0$ be func.s of x & y (these func.s can be complex, as usual).

$$\vec{E}_0 = E_{0x}\hat{x} + E_{0y}\hat{y} + E_{0z}\hat{z}$$

func.s of x, y

$$\vec{B}_0 = B_{0x}\hat{x} + B_{0y}\hat{y} + B_{0z}\hat{z}$$

For a rectangular pipe, the parallel (to the interior wall) components are $\hat{z}$ & $\hat{x}$ or $\hat{y}$ (depending on the surface), $\perp$ components are $\hat{x}$ or $\hat{y}$.

In an empty wave guide, one of the fields must have a longitudinal ($\hat{z}$) component.

To show - we cannot have both $E_{0z}=0$ & $B_{0z}=0$.

If both are zero, Maxwell's eqn.s give:

$$\nabla\times\vec{E}_0 = -\frac{\partial \vec{B}_0}{\partial t} \Rightarrow \frac{\partial E_{0y}}{\partial x} - \frac{\partial E_{0x}}{\partial y} = 0$$

so $\vec{E}_0 = -\nabla V$ (for $V(x,y)$, a $D=2$ potential)

&

$$\nabla\cdot\vec{E}_0 = 0 \Rightarrow \nabla^2 V = 0$$

the potential satisfies the Laplace eqn. inside the wave guide, the surface of the wave guide is an equipotential, & thus

$V = V_{surf.}$ $\forall\, x, y$ is the unique sol'n. to the Laplace problem

so

$\vec{E}_0 = 0$, no wave in the pipe.

=

Maxwell's Eqns

Inside the pipe, we have:

$$\nabla\cdot\vec{E}=0 \qquad \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \nabla\cdot\vec{B}=0 \qquad \nabla\times\vec{B} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$$

and for our setup in $(*)$, we have:

$$\frac{\partial\vec{E}}{\partial t} = -i\omega\vec{E},\quad \frac{\partial\vec{B}}{\partial t} = -i\omega\vec{B},\quad \frac{\partial\vec{E}}{\partial z} = ik\vec{E},\quad \frac{\partial\vec{B}}{\partial z} = ik\vec{B}$$

& we can rewrite Maxwell's eqns in terms of the $\vec{E}_0$ & $\vec{B}_0$ functions.

$$E_{0x} = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial E_{0z}}{\partial x} + \omega\frac{\partial B_{0z}}{\partial y}\right)$$

$$E_{0y} = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial E_{0z}}{\partial y} - \omega\frac{\partial B_{0z}}{\partial x}\right)$$

$$B_{0x} = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial B_{0z}}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_{0z}}{\partial y}\right)$$

$$B_{0y} = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial B_{0z}}{\partial y} + \frac{\omega}{c^2}\frac{\partial E_{0z}}{\partial x}\right)$$

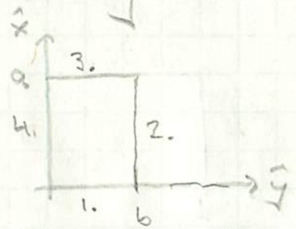
so that if you know E_{0z} & B_{0z}, you have everything else.

We also get eqn.s governing E_{0z} & B_{0z}:

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{c}\right)^2 - k^2\right]B_{0z} = 0 \quad (a) \text{ \& sim. for } E_{0z}.$$

<u>Transverse Electric ($E_{0z}=0$) Waves</u>

Take a wave guide w/ cross section:



& let $E_{0z} = 0$ - the boundary conditions are:

1. $E_{0y}(0,y) = 0$, $E_{0z}(0,y) = 0$, $B_{x}(0,y) = 0$
2. $E_{0x}(x,b) = 0$, $E_{0z}(x,b) = 0$, $B_{0y}(x,b) = 0$
3. $E_{0y}(a,y) = 0$, $E_{0z}(a,y) = 0$, $B_{0x}(a,y) = 0$
4. $E_{0x}(x,0) = 0$, $E_{0z}(x,0) = 0$, $B_{0y}(x,0) = 0$

Let's solve for B_{0z}, then see what the boundary conditions tell us. Use mult. S.O.V.: $B_{0z} = X(x)Y(y)$, then (a) is:

$$\underbrace{\frac{X''}{X}}_{=-k_x^2} + \underbrace{\frac{Y''}{Y}}_{=-k_y^2} + \left(\frac{\omega}{c}\right)^2 - k^2 = 0$$

so $X(x) = A\cos(k_x x) + B\sin(k_x x)$

$Y(y) = F\cos(k_y y) + G\sin(k_y y)$

& $k^2 = (\omega/c)^2 - k_x^2 - k_y^2$.

let's impose the 1. b.c.:

$$E_{0y}(0,y) = \frac{i}{(\omega/c)^2 - k^2}\left[-\omega X'(0)Y(y)\right] = 0 \Rightarrow B = 0$$

$$B_{0x}(0,y) = \frac{i}{(\omega/c)^2 - k^2}\left[kX'(0)Y(y)\right] = 0 \checkmark \quad (\text{w/ } B=0)$$

next, 4. $E_{0x}(x,0) = \frac{i}{(\omega/c)^2 - k^2}\left[\omega X(x)Y'(0)\right] = 0 \Rightarrow G = 0$

& then $B_{0y}(x,0) = 0$ ✓

so far, then, we have: $X(x) = A\cos(k_x x)$, $Y(y) = F\cos(k_y y)$
next, we'll impose 3.

$$E_{0y}(a,y) = \frac{i}{(\omega/c)^2 - k^2}\left[-\omega X'(a)\,Y(y)\right] = 0 \Rightarrow \sin(k_x a) = 0$$

so

$$k_x a = m\pi \Rightarrow k_x = m\pi/a$$

then $B_{0x}(a,y) = 0$ ✓

Finally, 2. has $E_{0x}(x,b) = 0 = B_{0y}(x,b)$
which require:

$$Y'(b) = 0 \Rightarrow \sin(k_y b) = 0 \Rightarrow k_y = n\pi/b.$$

The TE mode is:

$$B_{0z}(x,y) = A\cos(m\pi x/a)\cos(n\pi y/b)$$

w/ $k = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$

? what happens if

$$\omega^2 < c^2\pi^2\left[(m/a)^2 + (n/b)^2\right] \; ?$$

Imperfect Conductors

In an "Ohmic" material, or "imperfect" conductor, we have current flowing,

$$\vec{J} = \sigma\vec{E} \qquad (\sigma \to \infty \text{ for perfect conductors})$$

& Maxwell's eqn.s for a linear material w/ ϵ, μ, & conductivity σ, are

$$\nabla\cdot\vec{E} = \rho_f/\epsilon \qquad \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\cdot\vec{B} = 0 \qquad \nabla\times\vec{B} = \mu(\overbrace{\sigma\vec{E}}^{J_f}) + \mu\epsilon\frac{\partial\vec{E}}{\partial t}$$

We are not used to having ρ_f in a conductor.

charge conservation for the free charge says:

$$\frac{\partial\rho_f}{\partial t} = -\nabla\cdot\vec{J}_f = -\nabla\cdot(\sigma\vec{E}) = -\frac{\sigma}{\epsilon}\rho_f$$

so that $\rho_f(t) = e^{-\sigma/\epsilon\, t}\,\rho_{0f}$

the free charge dissipates to the surface. We should still consider the effect of the current $\sigma\vec{E}$ for plane waves impinging on these materials.