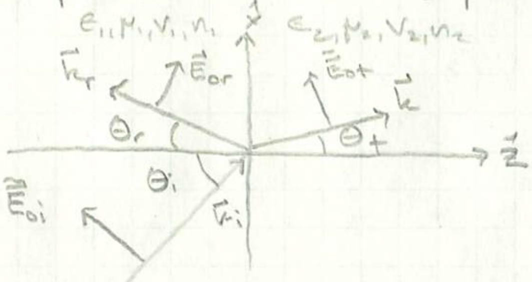
Fresnel Equations

For polarization in the x-z plane:



we have: $\tilde{\vec{E}}_i = \tilde{\vec{E}}_{0i} e^{i(\vec{k}_i \cdot \vec{r} - \omega t)}$

$\tilde{\vec{E}}_r = \tilde{\vec{E}}_{or} e^{i(\vec{k}_r \cdot \vec{r} - \omega t)}$

$\tilde{\vec{E}}_t = \tilde{\vec{E}}_{ot} e^{i(\vec{k}_t \cdot \vec{r} - \omega t)}$

& imposed: $\epsilon_1 E_1^\perp = \epsilon_2 E_2^\perp$ at $z=0$ (*)

$\vec{E}_1^\parallel = \vec{E}_2^\parallel$ at $z=0$ (o)

to get ✓: 1. $\vec{k}_i, \vec{k}_r, \vec{k}_t$ lie in a plane

2. $\theta_r = \theta_i$

3. $\sin\theta_t = \frac{n_1}{n_2}\sin\theta_i$

& using $\tilde{\vec{E}}_{0i} = \tilde{E}_{0i}(\cos\theta_i \hat{x} - \sin\theta_i \hat{z})$

$\tilde{\vec{E}}_{or} = \tilde{E}_{or}(\cos\theta_r \hat{x} + \sin\theta_r \hat{z})$

$\tilde{\vec{E}}_{ot} = \tilde{E}_{ot}(\cos\theta_t \hat{x} - \sin\theta_t \hat{z})$

i-(*): $\epsilon_1(-\tilde{E}_{0i}\sin\theta_i + \tilde{E}_{or}\sin\theta_r) = \epsilon_2 \tilde{E}_{ot}\sin\theta_t$

$\tilde{E}_{oi}\cos\theta_i + \tilde{E}_{or}\cos\theta_r = \tilde{E}_{ot}\cos\theta_t$

now from 2. & 3., these read:

$$(\tilde{E}_{oi} - \tilde{E}_{or}) = \underbrace{\frac{\epsilon_2}{\epsilon_1}\frac{n_1}{n_2}}_{=\frac{\mu_1 v_1}{\mu_2 v_2} \equiv \beta} \tilde{E}_{ot}$$

$$(\tilde{E}_{oi} + \tilde{E}_{or}) = \underbrace{\frac{\cos\theta_t}{\cos\theta_i}}_{\equiv \alpha} \tilde{E}_{ot}$$

add 'em: $2\tilde{E}_{oi} = (\alpha+\beta)\tilde{E}_{ot} \Rightarrow \tilde{E}_{ot} = \frac{2}{\alpha+\beta}\tilde{E}_{oi}$

subtract: $2\tilde{E}_{or} = (\alpha-\beta)\tilde{E}_{ot} = \frac{2(\alpha-\beta)}{\alpha+\beta}\tilde{E}_{ot}$

Fresnel eqns: $\tilde{E}_{or} = \frac{\alpha-\beta}{\alpha+\beta}\tilde{E}_{oi}$ $\quad \tilde{E}_{ot} = \frac{2}{\alpha+\beta}\tilde{E}_{oi}$

note: • $\tilde{E}_{ot}$ is in the same direction as $\tilde{E}_{oi}$, but $\tilde{E}_{or}$ could be opposite $\tilde{E}_{oi}$ — upside down.

• β is a constant, but $\alpha = \frac{\cos\theta_t}{\cos\theta_i}$ depends on θ_i

$$\alpha = \frac{\sqrt{1-\sin^2\theta_t}}{\cos\theta_i} = \frac{\sqrt{1-\left(\frac{n_1}{n_2}\sin\theta_i\right)^2}}{\cos\theta_i}$$

• as $\theta_i \to \pi/2$, $\alpha \to \infty$ & $\tilde{E}_{or} \to \tilde{E}_{oi}$

all the incident wave is reflected

• $\tilde{E}_{0r} = 0$ if $\alpha = \beta$, the incident angle is called "Brewster's angle"

$$1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_B = \beta^2\cos^2\theta_B = \beta^2(1-\sin^2\theta_B)$$

so

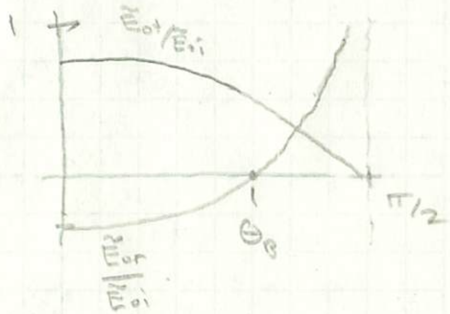
$$\sin^2\theta_B = \frac{\beta^2-1}{\beta^2-\left(\frac{n_1}{n_2}\right)^2}$$

• at $\theta_i = 0$, $\alpha = 1$ & $\tilde{E}_{0r} = \frac{1-\beta}{1+\beta}\tilde{E}_{0i}$

$$\tilde{E}_{0t} = \underbrace{\frac{2}{1+\beta}}_{\geq 0}\tilde{E}_{0i}$$

for $\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{n_2}{n_1}$ (μ's ≈ 1 often), & $n_2 > n_1$, $\tilde{E}_{0r} < 0$

take $n_1 = 1$, $n_2 > 1$ (air to water), the coefficient ratios can be sketched as a function of θ_i:



Intensities: $\vec{I} = \frac{1}{2}\epsilon v E^2 \hat{k}$ in general.

the intensity entering the interface from the incident wave is:

$$\vec{I}\cdot\hat{z} = \frac{1}{2}\epsilon_1 v_1 \tilde{E}_{0i}^2\cos\theta_i \equiv I_i \qquad \hat{z}\cdot\hat{k}_i = \cos\theta_i$$

the reflected intensity is:

$$\vec{I}_r\cdot\hat{z} = \frac{1}{2}\epsilon_1 v_1 \tilde{E}_{0r}^2\cos\theta_r = I_r$$

& the transmitted piece is

$$\vec{I}_t\cdot\hat{z} = \frac{1}{2}\epsilon_2 v_2 \tilde{E}_{0t}^2\cos\theta_t = I_t$$

we can take the ratios:

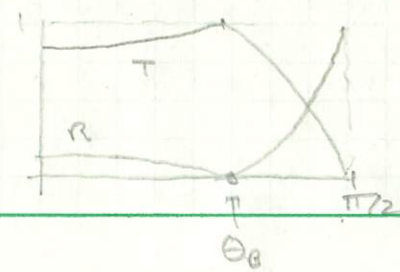
$$R = \frac{I_r}{I_i} = \frac{\tilde{E}_{0r}^2}{\tilde{E}_{0i}^2} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2$$

$$T = \frac{I_t}{I_i} = \frac{\epsilon_2 v_2 \tilde{E}_{0t}^2\cos\theta_t}{\epsilon_1 v_1 \tilde{E}_{0i}^2\cos\theta_i} = \alpha\beta\cdot\left(\frac{2}{\alpha+\beta}\right)^2$$

conservation of energy requires:

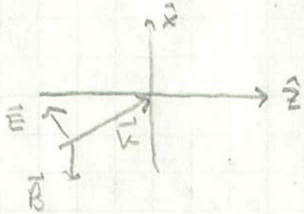
$R + T = 1$ check: $\frac{\alpha^2 - 2\alpha\beta + \beta^2 + 4\alpha\beta}{(\alpha+\beta)^2} = \frac{(\alpha+\beta)^2}{(\alpha+\beta)^2} = 1$ ✓

• a sketch of R, T as functions of θ_i looks like:

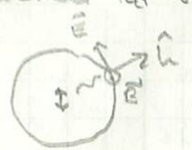


Wave Guides

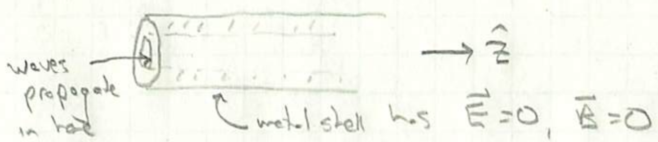
Our plane waves are infinite sheets:



we know that this is an approximation valid for radiation far from a dipole source:



We can localize plane waves using conducting boundaries:



the b.c. is, then: $\vec{E}^{\parallel} = 0$ at the surface

$B^{\perp} = 0$ " " "

(to keep the conductor an equipotential, surface current will move around, so we don't have $E^{\perp}=0$ or $\vec{B}^{\parallel}=0$ necessarily)

Assume we have a monochromatic plane wave travelling in the $\hat{z}$ direction.

$$\tilde{\vec{E}} = \tilde{\vec{E}}_0(x,y)e^{i(kz-\omega t)} \qquad \tilde{\vec{B}} = \tilde{\vec{B}}_0(x,y)e^{i(kz-\omega t)}$$

$\omega = kc$

w/ $\tilde{\vec{E}}_0 = E_x\hat{x} + E_y\hat{y} + E_z\hat{z}$, $\tilde{\vec{B}}_0 = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$

still complex in general — complex

the vacuum Maxwell's eqns are

$$\nabla\cdot\vec{E} = 0 \quad \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad \nabla\cdot\vec{B} = 0 \quad \nabla\times\vec{B} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} \qquad (*)$$

& we know the wave eqn. will emerge in the usual way — our assumed form already satisfies the wave eqn. ✓

How does the full set in (*) constrain the functions in $\tilde{\vec{E}}_0$ & $\tilde{\vec{B}}_0$?

$$\nabla\times\tilde{\vec{E}} = -\frac{\partial\tilde{\vec{B}}}{\partial t} \Longrightarrow \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{x} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{y} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{z}$$

$$= +i\omega\left[B_x\hat{x} + B_y\hat{y} + B_z\hat{z}\right]$$

& we get 3 eqns that can be used to relate E_x, B_x, E_y & B_y to E_z, B_z & their derivatives.

Then those 2 are governed by:

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2\right]E_z = 0$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2\right]B_z = 0$$

longitudinal components

If $E_z = 0$, these are "transverse electric" waves, if $B_z = 0$, they are "transverse magnetic" waves — if $E_z = 0$ & $B_z = 0$, they are "TEM" waves & these cannot occur for our setup.

since: $\nabla\cdot\vec{E} = 0$ & $\nabla\times\vec{E} = 0 \Rightarrow \nabla^2 V = 0$

& V has max/min only at the boundary — that boundary is an equipotential, so V has constant value throughout the cavity, & therefore $\vec{E} = 0$.