Linear Materials

For linear materials w/ $\vec{D} = \epsilon \vec{E}$, $\vec{H} = \frac{1}{\mu}\vec{B}$, Maxwell's eqn's read:

$$\nabla \cdot \vec{D} = \rho_f \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

& in free-charge "vacuum",

$$\nabla \cdot \vec{E} = 0 \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{B} = \underbrace{\mu\epsilon}_{=1/v^2} \frac{\partial \vec{E}}{\partial t} \qquad (*)$$

we set $n = c/v > 1$, the "index of refraction"

$$n = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} \approx \sqrt{\frac{\epsilon}{\epsilon_0}} = \epsilon_r \quad \text{"dielectric constant"}$$

$\mu \approx \mu_0$ for many materials.

plane wave solutions to the set in (*) have

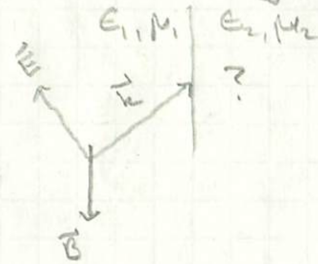
$$u = \tfrac{1}{2}\epsilon E^2 + \frac{1}{2\mu} B^2$$

$$\vec{S} = \frac{1}{\mu} \vec{E} \times \vec{B}$$

w/ "the usual" $B \sim \frac{1}{v} E$, so the intensity is:

$$I = \langle \vec{S} \rangle = \frac{1}{2}\frac{1}{\mu v} E^2 = \tfrac{1}{2}\epsilon v E^2$$

We will be interested in what happens to a wave that waves from one medium into another, & the boundary conditions will be important:

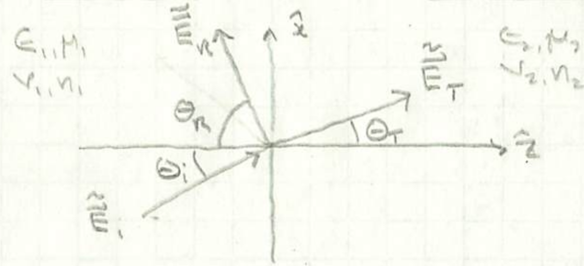


$$\epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp} \qquad \vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}$$

$$B_1^{\perp} = B_2^{\perp} \qquad \frac{1}{\mu_1}\vec{B}_1^{\parallel} = \frac{1}{\mu_2}\vec{B}_2^{\parallel}$$

Reflection & Transmission

The story: A plane wave comes in from the left, runs into a medium change, that gives rise to a reflected wave & a transmitted wave:



$$\tilde{\vec{E}}_i = \tilde{\vec{E}}_{0i}\, e^{i(\vec{k}_i \cdot \vec{r} - \omega t)} \qquad \text{w/} \quad k_i v_1 = \omega$$

$$\tilde{\vec{B}}_i = \frac{1}{v_1} \hat{k}_i \times \tilde{\vec{E}}_i$$

$$\tilde{\vec{E}}_r = \tilde{\vec{E}}_{0r}\, e^{i(\vec{k}_r \cdot \vec{r} - \omega t)}$$

$$\tilde{\vec{B}}_r = \frac{1}{v_1} \hat{k}_r \times \tilde{\vec{E}}_r$$

$$(k_r v_1 = \omega)$$

$$\tilde{\vec{E}}_T = \tilde{\vec{E}}_{0T}\, e^{i(\vec{k}_T \cdot \vec{r} - \omega t)}$$

$$\tilde{\vec{B}}_T = \frac{1}{v_2} \hat{k}_T \times \tilde{\vec{E}}_T$$

$$(k_T v_2 = \omega)$$

? Why is ω fixed ?

On the left, the total electric field is the sum of the insident + reflected fields.

The general structure of the boundary conditions is:

$$\_\_\_ e^{i(\vec{k}_i\cdot\vec{r}-\omega t)} + \_\_\_ e^{i(\vec{k}_r\cdot\vec{r}-\omega t)} = \_\_\_ e^{i(\vec{k}_t\cdot\vec{r}-\omega t)} \quad (*)$$

w/ $\vec{r} = x\hat{x} + y\hat{y}$. The $e^{-i\omega t}$ piece will cancel,

+

$$\vec{k}_i\cdot\vec{r} = k_{ix}x + k_{iy}y \qquad \vec{k}_t\cdot\vec{r} = k_{tx}x + k_{ty}y$$

$$\vec{k}_r\cdot\vec{r} = k_{rx}x + k_{ry}y$$

In order for (*) to hold $\forall$ x + y, we must have

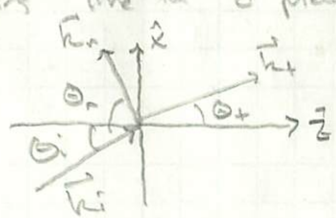
$$k_{ix} = k_{rx} = k_{tx} \quad + \quad k_{iy} = k_{ry} = k_{ty}$$

Suppose $\vec{k}_i$ lies in the xz plane, w/ $k_{iy}=0$, then

$$k_{ry} = 0 \quad + \quad k_{ty} = 0$$

so that $\vec{k}_r$ + $\vec{k}_t$ also lie in the xz plane

1. incident, reflected + transmitted wave vectors lie in a plane.



$$\vec{k}_i = k_{ix}\hat{x} + k_{iz}\hat{z}$$
$$= k_i\sin\theta_i\hat{x} + k_i\cos\theta_i\hat{z}$$

w/ $k_i v_1 = \omega$

and $\vec{k}_r = k_{rx}\hat{x} + k_{rz}\hat{z}$

$$= k_r\sin\theta_r\hat{x} - k_r\cos\theta_r\hat{z}$$

w/ $k_r v_1 = \omega \Rightarrow k_i = k_r$, + so

2. $\sin\theta_i = \sin\theta_r$ angle of incidence equals angle of reflection
$\Downarrow$
$\theta_i = \theta_r$.

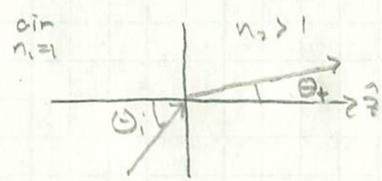
we also have $\vec{k}_t = k_t\sin\theta_t\hat{x} + k_t\cos\theta_t\hat{z}$
+ setting $k_{tx} = k_{ix}$ gives

3. $$\frac{\sin\theta_t}{\sin\theta_i} = \frac{k_i}{k_t} = \frac{1/v_1}{1/v_2} = \frac{v_2}{v_1} = \frac{n_1}{n_2}$$ Snell's Law.

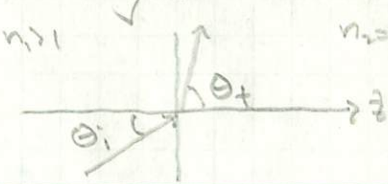
If a wave goes from air w/ $n_1=1$, to another medium w/ $n_2 > 1$,

$$\sin\theta_t = \frac{1}{n_2}\sin\theta_i \Rightarrow \theta_t < \theta_i$$

the wave bends <u>towards</u> the normal:



If a wave goes from $n_1 > 1$ to air; $n_2 = 1$, $\theta_t > \theta_i$



note that in this case, there exists an incident angle $\theta_i = \Theta_c$ that leads to a transmitted angle of $\Theta_t = \pi/2$:

$$\sin\Theta_t = n_1 \sin\Theta_i \Rightarrow \sin\Theta_c = \frac{1}{n_1}$$

? What happens after that?

<u>Boundary Conditions</u>

$$\epsilon_1[\tilde{\vec{E}}_{0i} + \tilde{\vec{E}}_{0r}]_z = \epsilon_2[\tilde{\vec{E}}_{0t}]_z \qquad (\perp)$$

$$[\tilde{\vec{E}}_{0i} + \tilde{\vec{E}}_{0r}]_{x,y} = [\tilde{\vec{E}}_{0t}]_{x,y} \qquad (\parallel)$$

$$[\tilde{\vec{B}}_{0i} + \tilde{\vec{B}}_{0r}]_z = [\tilde{\vec{B}}_{0t}]_z \qquad (\perp)$$

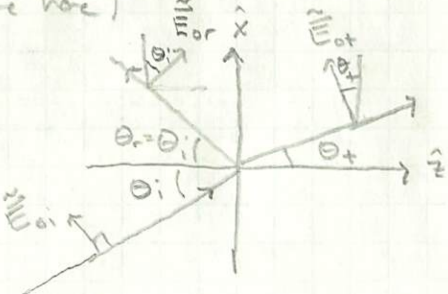
$$\frac{1}{\mu_1}[\tilde{\vec{B}}_{0i} + \tilde{\vec{B}}_{0r}]_{x,y} = \frac{1}{\mu_2}[\tilde{\vec{B}}_{0t}]_{x,y} \qquad (\parallel)$$

w/

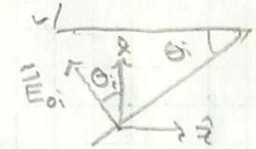
$$\tilde{\vec{B}}_{0i} = \frac{1}{v_1}\hat{k}_i \times \tilde{\vec{E}}_{0i}, \quad \tilde{\vec{B}}_{0r} = \frac{1}{v_1}\hat{k}_r \times \tilde{\vec{E}}_{0r}$$

$$\tilde{\vec{B}}_{0t} = \frac{1}{v_2}\hat{k}_t \times \tilde{\vec{E}}_{0t}$$

consider the case in which the polarization vector is in the plane of incidence (the xz plane here)



w/



$$\tilde{\vec{E}}_{0i} = \tilde{E}_{0i}(\cos\theta_i \hat{x} - \sin\theta_i \hat{z})$$

and sim.

$$\tilde{\vec{E}}_{0r} = \tilde{E}_{0r}[\cos\theta_i \hat{x} + \sin\theta_i \hat{z}]$$

$$\tilde{\vec{E}}_{0t} = \tilde{E}_{0t}[\cos\theta_t \hat{x} - \sin\theta_t \hat{z}]$$

then the $\perp$ eqn. for $\vec{E}$ gives:

$$\epsilon_1(-\tilde{E}_{0i}\sin\theta_i + \tilde{E}_{0r}\sin\theta_i) = -\epsilon_2\tilde{E}_{0t}\sin\theta_t$$

& the $\parallel$ eqn. for $\vec{E}$ says:

$$\tilde{E}_{0i}\cos\theta_i + \tilde{E}_{0r}\cos\theta_i = \tilde{E}_{0t}\cos\theta_t$$

the $\parallel$ $\vec{B}$ eqn. yields

$$\frac{1}{\mu_1 v_1}(\tilde{E}_{0i} - \tilde{E}_{0r}) = \frac{1}{\mu_2 v_2}\tilde{E}_{0t}$$

let $\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2}$ $\quad \alpha \equiv \frac{\cos\theta_t}{\cos\theta_i}$

then $\tilde{E}_{0r} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)\tilde{E}_{0i}$ $\quad \tilde{E}_{0t} = \frac{2}{\alpha+\beta}\tilde{E}_{0i}$

the Fresnel eqn.s