Momentum Conservation

we suggested that fields might carry momentum, + even found a candidate "field momentum density"

$$\vec{g} = \frac{1}{c^2}\vec{S} = \epsilon_0 \vec{E} \times \vec{B}.$$

let's think about what conservation of field momentum might look like in an empty (of charges) region Ω:

$$\frac{d}{dt}\int_\Omega \vec{g}\, d\tau = -\oint_{\partial\Omega} \underline{?} \cdot d\vec{a} \quad (*)$$

? is some sort of object that, when dotted into a vector ($d\vec{a}$) returns a <u>vector</u> (the LHS of (*) is a vector)

A <u>matrix</u> can be "dotted into a vector" returning a vector. Matrix vector multiplication says:

$$\vec{w} = \mathbb{A}\vec{v} \quad w/ \quad w^i = \sum_{j=1}^{3} A^{ij} v^j \ (i = 1,2,3)$$

sometimes written: $\vec{w} = \overleftrightarrow{A} \cdot \vec{v}$

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Last time we developed the combined particle-field

$$\frac{dW}{dt} = -\frac{d}{dt}\int_\Omega u\, d\tau - \oint_{\partial\Omega} \vec{S} \cdot d\vec{a}$$

work done on the charges in Ω

a statement of energy conservation

today we want to express the similar "momentum" conservation combining particles w/ fields:

change in mechanical momentum

$$\frac{d\vec{p}}{dt} = -\frac{d}{dt}\int_\Omega \vec{g}\, d\tau - \oint_{\partial\Omega} \underline{?} \cdot d\vec{a}$$

To start off take a domain Ω w/ some individual charges

the force on each charge is

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

moving to a continuum description: $q \longrightarrow \rho\, d\tau$

$$\vec{F} = \int_\Omega (\rho\vec{E} + \underbrace{\rho\vec{v}}_{=\vec{J}}\times\vec{B})\, d\tau = \int_\Omega (\rho\vec{E} + \vec{J}\times\vec{B})\, d\tau$$

w/ force <u>density</u> $\vec{f} = \rho\vec{E} + \vec{J}\times\vec{B}$. The $\vec{F}$ is a good starting point since

$$\vec{F} = \frac{d\vec{p}}{dt} \quad \text{where } \vec{p} \text{ is the particle or "mechanical" momentum.}$$

we'll again focus on $\vec{f}$, replacing ρ + $\vec{J}$ w/ their expressions in terms of $\vec{E}$ + $\vec{B}$ + their various derivatives.

$$\vec{f} = \rho\vec{E} + \vec{J}\times\vec{B} \quad w/ \quad \rho = \epsilon_0 \nabla\cdot\vec{E}, \ \vec{J} = \frac{1}{\mu_0}\nabla\times\vec{B} - \epsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$= \epsilon_0(\nabla\cdot\vec{E})\vec{E} + \frac{1}{\mu_0}(\nabla\times\vec{B})\times\vec{B} - \epsilon_0\frac{\partial\vec{E}}{\partial t}\times\vec{B}$$

targeting familiar quantities, like $\vec{E}\times\vec{B}$ ($\sim\vec{S}$)

$$\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) = \frac{\partial \vec{E}}{\partial t}\times\vec{B} + \vec{E}\times\underbrace{\frac{\partial\vec{B}}{\partial t}}_{= -\nabla\times\vec{E}}$$

so that:
$$\frac{\partial\vec{E}}{\partial t}\times\vec{B} = \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) + \vec{E}\times(\nabla\times\vec{E})$$

giving

$$\vec{f} = \epsilon_0[(\nabla\cdot\vec{E})\vec{E} - \vec{E}\times(\nabla\times\vec{E})] + \frac{1}{\mu_0}[\underbrace{(\nabla\cdot\vec{B})}_{=0,\text{ add in for aesthetics}}\vec{B} - \vec{B}\times(\nabla\times\vec{B})] - \epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

Finally, we can use the identity:
$$\vec{E}\times(\nabla\times\vec{E}) = \tfrac{1}{2}\nabla(E^2) - (\vec{E}\cdot\nabla)\vec{E}$$
& sim for $\vec{B}$:

$$\vec{f} = \epsilon_0[(\nabla\cdot\vec{E})\vec{E} + (\vec{E}\cdot\nabla)\vec{E} - \tfrac{1}{2}\nabla(E^2)]$$
$$+\frac{1}{\mu_0}[(\nabla\cdot\vec{B})\vec{B} + (\vec{B}\cdot\nabla)\vec{B} - \tfrac{1}{2}\nabla(B^2)] - \frac{\partial}{\partial t}\overbrace{(\epsilon_0\vec{E}\times\vec{B})}^{\equiv\vec{g}\text{ momentum density}}$$

Define: $T_{ij} = \epsilon_0(E_iE_j - \tfrac{1}{2}\delta_{ij}E^2) + \frac{1}{\mu_0}(B_iB_j - \tfrac{1}{2}\delta_{ij}B^2)$

then
$$\sum_{i=1}^{3}\frac{\partial T_{ij}}{\partial x^i} = \sum_{i=1}^{3}\epsilon_0\left(\frac{\partial E^i}{\partial x^i}E_j + E^i\frac{\partial E_j}{\partial x^i}\right) - \frac{1}{2}\frac{\partial(E^2)}{\partial x^j} + \cdots$$
← for $j = 1, 2, 3$

sim. for the B terms

$$\nabla\cdot\overleftrightarrow{T} = \epsilon_0(\vec{E}(\nabla\cdot\vec{E}) + (\vec{E}\cdot\nabla)\vec{E} - \tfrac{1}{2}\nabla(E^2))$$
$$+\frac{1}{\mu_0}(\vec{B}(\nabla\cdot\vec{B}) + (\vec{B}\cdot\nabla)\vec{B} - \tfrac{1}{2}\nabla(B^2))$$
$$= \nabla\cdot\overleftrightarrow{T}$$

and we can write:
$$\vec{f} = -\nabla\cdot(-\overleftrightarrow{T}) - \frac{\partial}{\partial t}\vec{g}$$

& going back to $\frac{d\vec{p}}{dt} = \vec{F}$ w/ $\vec{F} = \int_\Omega \vec{f}\,d\tau$,

$$\frac{d\vec{p}}{dt} = -\int_\Omega\frac{\partial}{\partial t}\vec{g}\,d\tau - \int_\Omega\nabla\cdot(-\overleftrightarrow{T})\,d\tau$$
$$= -\frac{d}{dt}\int_\Omega\vec{g}\,d\tau - \oint_{\partial\Omega}(-\overleftrightarrow{T})\cdot d\vec{a} \quad (*)$$

the momentum analogue of

$$\frac{dW}{dt} = -\frac{d}{dt}\int_\Omega u\,d\tau - \oint_{\partial\Omega}\vec{S}\cdot d\vec{a} \quad (+)$$

for energy.

Note that $\vec{S}$ appears as an energy current density in (+), & as the conserved $\vec{g} = \epsilon_0\vec{S}$ in (*)