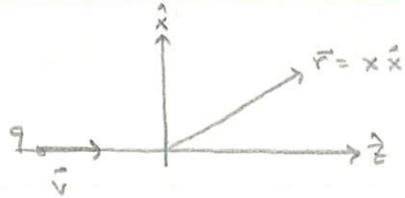
Potentials for Constant Velocity Motion



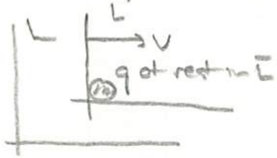
$\vec{w}(t) = vt\,\hat{z}$
$\dot{\vec{w}}(t) = v\,\hat{z}$
$\vec{r} = x\,\hat{x} + z\,\hat{z}$.

we thought about using $V(\vec{r},t) = \dfrac{q}{4\pi\epsilon_0}\dfrac{1}{(\mathscr{r} - \vec{\mathscr{r}}\cdot\vec{v}/c)}$

w/ $\vec{\mathscr{r}} = \vec{r} - \vec{w}(t_r)$ $\qquad \vec{A}(\vec{r},t) = \dfrac{\mu_0 q}{4\pi}\dfrac{\vec{v}}{(\mathscr{r} - \vec{\mathscr{r}}\cdot\vec{v}/c)}$ $\leftarrow = v\hat{z}$
$\vec{v} = \dot{\vec{w}}(t_r)$

& $c(t - t_r) = |\vec{r} - \vec{w}(t_r)| \Rightarrow c^2(t-t_r)^2 = (x^2 + (z - vt_r)^2)$

but decided to use our SR knowledge:

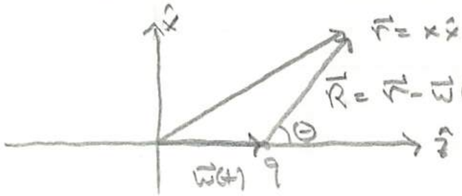


$$\left.\begin{aligned}\bar{V} &= \frac{q}{4\pi\epsilon_0}\frac{1}{(\bar{x}^2 + \bar{z}^2)^{1/2}} \\ \bar{A}_z &= 0\end{aligned}\right\} \text{ in } \bar{L}$$

in L: $V = \gamma\bar{V}$, $A_z = \gamma\, v/c^2\, \bar{V}$

$$V = \frac{q}{4\pi\epsilon_0}\frac{1}{(x^2(1-v^2/c^2) + (z-vt)^2)^{1/2}} \qquad \vec{A} = \frac{\mu_0 q}{4\pi}\frac{v\hat{z}}{(x^2(1-v^2/c^2) + (z-vt)^2)^{1/2}}$$

It is interesting to note that we can write V & $\vec{A}$ in terms of the instantaneous position of the charge:



then $R^2 = x^2 + (z - vt)^2$

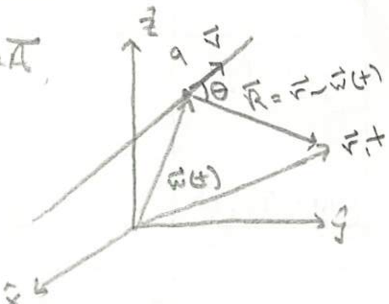
& $x^2(1 - v^2/c^2) + (z - vt)^2 = R^2 - v^2/c^2\, x^2$ w/ $x = R\sin\theta$
$= R^2(1 - v^2/c^2 \sin^2\theta)$.

so $V(\vec{r},t) = \dfrac{q}{4\pi\epsilon_0 R}\dfrac{1}{\sqrt{1 - v^2/c^2\sin^2\theta}}$.

$\vec{A} = \dfrac{\mu_0 q}{4\pi R}\dfrac{\vec{v}}{\sqrt{1 - v^2/c^2\sin^2\theta}}$.

Now it's not too bad to calculate:

$\vec{E} = -\nabla V - \dfrac{\partial\vec{A}}{\partial t}$, $\vec{B} = \nabla\times\vec{A}$.



& you get:

$\vec{E} = \dfrac{q\hat{R}}{4\pi\epsilon_0 R^2}\dfrac{(1 - v^2/c^2)}{(1 - v^2/c^2\sin^2(\theta))^{3/2}}$

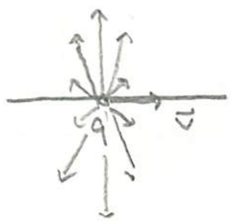
$\vec{B} = \dfrac{\mu_0 q}{4\pi R^2}\dfrac{(\vec{v}\times\hat{R})(1 - v^2/c^2)}{(1 - v^2/c^2\sin^2(\theta))^{3/2}} = \dfrac{1}{c^2}\vec{v}\times\vec{E}$

as the fields for a particle moving w/ constant velocity $\vec{v}$ (Heaviside fields)

remember that $\vec{R} = \vec{r} - \vec{w}(t)$ here, so it's the vector pointing from the current (!) particle location to the field point

For $v/c \ll 1$, $\vec{E} = \dfrac{q}{4\pi\epsilon_0 R^2}\hat{R}$ $\quad \vec{B} = \dfrac{\mu_0 q}{4\pi}\dfrac{\vec{v}\times\hat{R}}{R^2}$

What do these fields look like? At what θ is $\vec{E}, \vec{B}$ the largest?



$1 - \frac{v^2}{c^2} \sin^2\theta$ smallest for $\theta = \pi/2$

Aside: For the energy-momentum 4-vec: $p^\mu \doteq \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix}$ in the rest frame of a particle, we have:

$\bar{E} = mc^2$

$\bar{p}_x = 0$

& in L: $E = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$, $p_x = \frac{mv}{\sqrt{1 - v^2/c^2}}$

just as $-c^2 \bar{t}^2 + \bar{x}^2 = -c^2 t^2 + x^2$, we have

$$-\bar{E}^2/c^2 + \bar{p}_x^2 = -E^2/c^2 + p_x^2$$

& in the rest frame, $-\bar{E}^2/c^2 + \bar{p}_x^2 = -m^2 c^2$

we expect: $-E^2/c^2 + p_x^2 = -m^2 c^2$, too, of course

let's check:

$$-\frac{m^2 c^2}{(1 - v^2/c^2)} + \frac{m^2 v^2}{(1 - v^2/c^2)} = \frac{-m^2 c^2 (1 - v^2/c^2)}{(1 - v^2/c^2)} = -m^2 c^2 \checkmark$$

in general: $-E^2/c^2 + \vec{p} \cdot \vec{p} = -m^2 c^2$

for massless particles, then,

$$E^2 = \vec{p} \cdot \vec{p}\, c^2 + 0$$

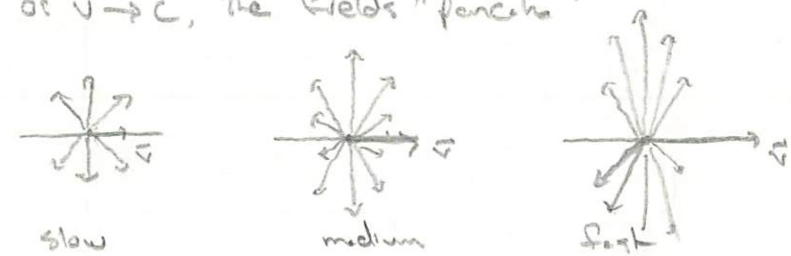
But if E is to have finite value for $m \to 0$:

$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$ as $m \to 0$, we must have $v \to c$

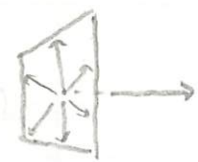
"massless particles travel at c"

Let's go back to our picture for $\vec{E}$ generated by a charge moving w/ constant velocity:

as $v \to c$, the fields "pancake"



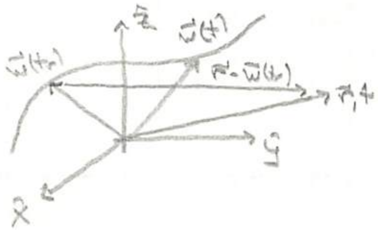
$\vec{E} \to \frac{q}{\epsilon_0} \frac{\delta(s - ct)}{s} \hat{s}$



$\vec{v}$ w/ $v = c$

this would be the electric field of a massless charged particle.

$\vec E$ + $\vec B$ for a Point Particle



we had the LW potentials:

$$\vec{\mathscr{r}} = \vec r - \vec w(t_r) \quad \text{w/}$$

$$c(t-t_r) = |\vec r - \vec w(t_r)|$$

+ $\vec v = \dot{\vec w}(t_r)$, then $V(\vec r,t) = \frac{q}{4\pi\epsilon_0}\frac{1}{\mathscr{r} - \vec{\mathscr{r}}\cdot\vec v/c}$

$$\vec A(\vec r,t) = \frac{\mu_0 q}{4\pi}\frac{\vec v}{\mathscr{r} - \vec{\mathscr{r}}\cdot\vec v/c}$$

and now it's time to compute the fields:

$$\vec E = -\nabla V - \frac{\partial \vec A}{\partial t} \qquad \vec B = \nabla \times \vec A.$$

The difficulty is in evaluating the derivatives of t_r.

example: $c(t-t_r) = |\vec r - \vec w(t_r)|$ take $\frac{\partial}{\partial t}$ of both sides

$$c\left(1 - \frac{\partial t_r}{\partial t}\right) = -\frac{(\vec r - \vec w(t_r))\cdot \frac{d\vec w}{dt_r}\frac{\partial t_r}{\partial t}}{|\vec r - \vec w(t_r)|}$$

+ use the notation defined above:

$$c\left(1 - \frac{\partial t_r}{\partial t}\right) = -\hat{\mathscr{r}}\cdot\vec v\,\frac{\partial t_r}{\partial t}$$

$$\frac{\partial t_r}{\partial t}\left(1 - \frac{\hat{\mathscr{r}}\cdot\vec v}{c}\right) = 1 \Rightarrow \frac{\partial t_r}{\partial t} = \frac{1}{1 - \hat{\mathscr{r}}\cdot\vec v/c}$$

and sim. $\nabla t_r = -\frac{\hat{\mathscr{r}}/c}{1 - \hat{\mathscr{r}}\cdot\vec v/c}$

Putting all the pieces together, we have, letting

$$u \equiv c\hat{\mathscr{r}} - \vec v \quad + \quad \vec a \equiv \ddot{\vec w}(t_r)$$

$$\vec E(\vec r,t) = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\vec u\cdot\vec{\mathscr{r}})^3}\left[(c^2 - v^2)\vec u + \vec{\mathscr{r}}\times(\vec u\times\vec a)\right]$$

$$\vec B = \frac{1}{c}\hat{\mathscr{r}}\times\vec E$$